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Quasimartingales on Partially Ordered Sets*

HARRY E. HÜRZELER

ETH-Zentrum, Zürich, Switzerland

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The concept of a quasimartingale, and therefore also of a function of bounded variation, is extended to processes with a regular partially ordered index set V and with values in a Banach space. We show that quasimartingales can be described by their associated measures, defined on an inverse limit space $S \times \Omega$ containing $V \times \Omega$, furnished with the σ-algebra $\mathscr{P}$ of the predictable sets. With the help of this measure, a Rao–Krickeberg and a Riesz decomposition is obtained, as well as a convergence theorem for quasimartingales. For a regular quasimartingale X it is proven that the spaces $(S \times \Omega, \mathscr{P})$ and the measures associated with X are unique up to isomorphisms. In the case $V = \mathbb{R}_+^n$ we prove a duality between classical (right-) quasimartingales and left-quasimartingales.

0. Introduction

A leftcontinuous function f on $\mathbb{R}$ is of bounded variation iff there exists a measure μ^f on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that $f(s) = \mu^f\{x \geqslant s\}$. The stochastic analogue has been given by Doléans-Dade [6] and Föllmer [8, 9]: under some regularity conditions, a process X is a quasimartingale iff there exists a measure P^X, sometimes called the Doléans or Föllmer measure of X, defined on the predictable sets $\mathscr{P}$ such that $P^X[(s, \infty] \times F] = E[X_s; F]$, $\forall F \in \mathscr{F}_s$. These results have been proven in the case where the index set is $\mathbb{R}_+^n$ (see [3, 10]), and the aim of this paper is to extend them further to partially ordered sets V satisfying some mild conditions.

The basic idea, suggested to me by Föllmer, is to use the Möbiusinversion: for if V is any finite partially ordered set and f is a function on V, then this inversion determines a function g such that $f(s) = \sum_{t \geqslant s} g(t)$; therefore the measure μ^f defined by $\mu^f(A) = \sum_{t \in A} g(t)$ is precisely what we are looking for.

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The second step is to construct the measure μ^f if V is not finite anymore, but "exhaustible." V can then be partitioned into a finite number of disjoint "rectangles" for which μ^f can be defined by the above method. Refining the partitioning, μ^f is determined on an increasing sequence $(\mathscr{K}_i)_{i\in N}$ of σ-algebras, and this leads to the third step, namely, the extension of μ^f to a measure on $\sigma(\bigcup_{i\in N}\mathscr{K}_i)$: since it is possible that there exist decreasing sequences of rectangles $R_i\in\mathscr{K}_i$ with $R_i\downarrow\varnothing$, $\mu^f(R_i)\not\to 0$, we have to add an element to V for every such sequence to get a larger space S; then, if f is of "bounded variation," μ^f can be extended to a measure on S, due to results of Parthasarathy [19]. The construction of this space S is done in Section 1, whereas the construction of the measure μ^f is presented in Section 2.

The last step (Section 4) is to work with a partially ordered set having an exhaustible dense set V'. Under some continuity assumptions the results then follow immediately from the analogous results on V'.

As an immediate application we get some decomposition theorems, such as a generalization of the Rao decomposition (Section 3). Other applications in the theory of stochastic integration, in the theory of optional stopping and for the Doob decomposition on partially ordered sets will be treated elsewhere (see, e.g., [12]).

Our construction of the measure P^X associated with the quasimartingale X on V differs from the ones in [3, 6, 8–10] in two basic points. First, the predictable sets $\mathscr{P}$ on which P^X is defined are generated by $[s,\infty[\times F_s$ rather than $(s,t]\times F_s$; this definition is necessary since in most interesting situations the rectangles in $\mathscr{K}_i$ do not have a sensible lowest vertex s. However, in the classical case $V=\mathbb{R}_+$, both approaches are equivalent as is shown in Section 5 since in fact one follows from the other by changing from right continuous to leftcontinuous modifications and vice versa. The second point is that P^X depends on the chosen dense set V'; but in Section 6 we show that the measure spaces $(S\times\Omega,\mathscr{P},P^X)$ are isomorphic (in the sense of [11]) for different V'.

Since this article addresses itself to probabilists, we have been very explicit whenever combinatorial arguments have appeared. The few results needed from the theory of combinatorics have all been summed up in the Appendix.

1. The Inverse Limit Space of Exhaustible Sets

To explain the term "inverse limit space," let V be any partially ordered set and $\mathbf{K}=(K_i)_{i\in\mathbb{N}}$ be an increasing sequence of finite subsets K_i of V. By setting

$$\mathscr{K}_i:=\sigma\{\{x\in V\mid x\geqslant t\}\mid t\in K_i\},$$

$\mathbf{K}$ defines an increasing sequence of σ-algebras $(\mathscr{K}_i)_{i\in\mathbb{N}}$ on V, such that $\mathscr{K}_i$

contains only finitely many sets. Therefore $(V, \mathscr{K}_i)$ is standard Borel in the sense of Parthasarathy [19]. Now for any sequence $(\Omega, \mathscr{A}_i)_{i \in \mathbb{N}}$ of standard Borel spaces with $\mathscr{A}_i \subseteq \mathscr{A}_{i+1}$, $\forall i \in \mathbb{N}$, we define the measurable space $(J, \mathscr{T})$ by

$$J := \{(A_i)_{i \in \mathbb{N}} \mid A_i \text{ is an atom in } (\Omega, \mathscr{A}_i), A_i \supseteq A_{i+1} \, \forall i \in \mathbb{N}\};$$

$$\mathscr{T} := \sigma\{\{(A_i)_{i \in \mathbb{N}} \in J \mid A_n = B\} \mid B \in \mathscr{A}_n, n \in \mathbb{N}\}.$$

$(J, \mathscr{T})$ is called the inverse limit space associated with the sequence $(\Omega, \mathscr{A}_i)_{i \in \mathbb{N}}$. Its importance lies in the following theorem of Parthasarathy [19], which we cite for the convenience of the reader:

THEOREM 1.1. *Let $(\Omega, \mathscr{A}_i)_{i \in \mathbb{N}}$ be a sequence of standard Borel spaces with $\mathscr{A}_i \subseteq \mathscr{A}_{i+1}$, $\forall i \in N$, and let μ_i be a signed measure on $(\Omega, \mathscr{A}_i)$ such that $(\mu_i)_{i \in \mathbb{N}}$ is a consistent sequence of measures with* $\sup_{i \in \mathbb{N}} \|\mu_i\| < \infty$. *Then there exists a unique finite measure μ on the inverse limit space $(J, \mathscr{T})$ associated with $(\Omega, \mathscr{A}_i)_{i \in \mathbb{N}}$, such that $\mu \circ \Pi_i^{-1} = \mu_i$, $\forall i \in \mathbb{N}$, where $\Pi_n((A_i)_{i \in \mathbb{N}}) := A_n$. In particular, if $\bigcap_{i \in \mathbb{N}} A_i \neq \emptyset$ whenever $(A_i)_{i \in \mathbb{N}}$ is a decreasing sequence of sets such that A_i is an atom in $(\Omega, \mathscr{A}_i)$, there exists a measure $\bar{\mu}$ on $(\Omega, \sigma(\bigcup_{i \in \mathbb{N}} \mathscr{A}_i))$ such that $\bar{\mu}|_{\mathscr{A}_i} = \mu_i$, $\forall i \in \mathbb{N}$; in fact, $(J, \mathscr{T})$ and $(\Omega, \sigma(\bigcup_{i \in \mathbb{N}} \mathscr{A}_i))$ are σ-isomorph.*

In [19] the theorem is only formulated for probability measures $(\mu_i)_{i \in \mathcal{N}}$. But our situation is easily reduced to this case by noticing that $(\mu_i)_{i \in \mathbb{N}}$ is a martingale of bounded variation in the sense of Krickeberg [13], and then applying the Krickeberg decomposition to $(\mu_i)_{i \in \mathbb{N}}$.

As we will be defining precisely such a consistent sequence of measures $(\mu_i)_{i \in \mathbb{N}}$ on $(V, \mathscr{K}_i)_{i \in \mathbb{N}}$ in the next paragraph, we are interested in a characterization of the inverse limit space $(J, \mathscr{T})$ associated with this sequence $(V, \mathscr{K}_i)_{i \in \mathbb{N}}$. For this purpose we have to discern two cases: if $V = \bigcup_{t \in K_n} \{x \in V \mid x \geqslant t\}$ for an $n \in \mathbb{N}$, we define $(J(\mathbf{K}), \mathscr{T}(\mathbf{K})) := (J, \mathscr{T})$; if there exists no $n \in \mathbb{N}$ with $V = \bigcup_{t \in K_n} \{x \in V \mid x \geqslant t\}$, then $A_i := V \setminus \bigcup_{t \in K_i} \{x \in V \mid x \geqslant t\}$ is an atom in $\mathscr{K}_i$ such that $(A_i)_{i \in \mathbb{N}}$ belongs to J, and we define $(J(\mathbf{K}), \mathscr{T}(\mathbf{K})) := (J \setminus \{(A_i)_{i \in \mathbb{N}}\}, \mathscr{T}|_{J(\mathbf{K})})$.

DEFINITION 1.2. $(J(\mathbf{K}), \mathscr{T}(\mathbf{K}))$ is called the inverse limit space associated with $\mathbf{K}$ on V.

To be able to define the measures μ_i on $(V, \mathscr{K}_i)$ we need some regularity conditions on the space V and the sequence $\mathbf{K}$; in addition, we will assume V to be countable for the time being, a condition which we will drop later.

DEFINITION 1.3. A countable partially ordered set V is called an exhaustible poset if there exists an increasing sequence $\mathbf{K} = (K_i)_{i \in \mathbb{N}}$ of finite sets $K_i \subseteq V$ such that

(i) $V = \bigcup_{i \in \mathbb{N}} K_i$;

(ii) K_i is $\bigcap$-compatible $\forall i \in \mathbb{N}$, i.e., $\forall r, s \in K_i$ there exist integers c_t such that

$$\mathbf{1}_{\{x \in V \mid x \geqslant r\} \cap \{x \in V \mid x \geqslant s\}} = \sum_{t \in K_i} c_t \, \mathbf{1}_{\{x \in V \mid x \geqslant t\}}.$$

Such a sequence $\mathbf{K}$ is called regularly exhausting.

Examples of exhaustible sets.

(1) $V = D^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_j \in \{k2^{-i}, k, i \in \mathbb{N}\},\ j = 1, ..., n\}$ the dyadic numbers, with $K_i = D_i^n := \{(x_1, ..., x_n) \in V \mid x_j \in \{k2^{-i} \mid 0 \leqslant k \leqslant i2^i\}\}$; or, similarly, $V = \mathbb{Q}_+^n$.

(2) A countable basis V of a separable topological space T, with the partial order defined by inclusion, and such that if A and B belong to $V, A \cup B$ is also in V. If $V = (U_j)_{j \in \mathbb{N}}$, then $K_i := \{\bigcup_{j \in I} U_j \mid I \subseteq \{1, ..., i\}\}$ defines a regularly exhausting sequence.

(3) A countable $\bigvee$-semilattice V, i.e., a poset such that $\forall s, t \in V$ there exists the lowest upper bound $s \vee t$. If $V = (s_j)_{j \in \mathbb{N}}$, then $K_i := \{\bigvee_{j \in I} s_j \backslash I \subseteq \{1, ..., i\}\}$ defines a regularly exhausting sequence.

(4) A countable tree V, i.e., $\forall s, t \in V$ we either have $s \geqslant t, t \geqslant s$ or $\{x \in V \mid x \geqslant s\} \cap \{x \in V \mid x \geqslant t\} = \varnothing$. If $V = (s_j)_{j \in \mathbb{N}}, K_i := \{s_1, ..., s_i\}$ gives a regularly exhausting sequence. ∎

If there exists an $n \in \mathbb{N}$ such that $V = \bigcup_{t \in K_n} \{x \in V \mid x \geqslant t\}$, we will always assume $n = 1$. This is no loss of generality since we could work with the sequence $(\bar{K}_i)_{i \in \mathbb{N}} := (K_{i+n})_{i \in \mathbb{N}}$.

So let V be an exhaustible poset with regularly exhausting sequence $\mathbf{K} = (K_i)_{i \in \mathbb{N}}$ and associated inverse limit space $(J(\mathbf{K}), \mathscr{T}(\mathbf{K}))$. Then the main characterization of $(J(\mathbf{K}), \mathscr{T}(\mathbf{K}))$ is given by Theorem 1.4 and Lemma 1.6 below:

THEOREM 1.4. *There exists a partial order $\geqslant_J$ on $J(K)$ and an embedding $v: V \to J(K)$ such that v is order-preserving, i.e., $s \geqslant t \Leftrightarrow v(s) \geqslant_J v(t)$, $\forall s, t \in V$, and such that $\mathscr{T}(\mathbf{K}) = \sigma\{\{x \in J(\mathbf{K}) \mid x \geqslant_J t\} \mid t \in J(\mathbf{K})\}$.*

The proof of the theorem will be split into a number of lemmas.

For $i \in \mathbb{N}$ and A an atom of $\mathscr{K}_i$, define $\varphi_i(A)$ as the greatest lower bound of A in V, if it exists.

LEMMA 1.5. *$\forall i \in \mathbb{N}$ and every atom A in $\mathscr{K}_i$ with $A \subseteq \bigcup_{t \in K_i}\{x \in V | x \geqslant t\}$ $\varphi_i(A)$ is defined, and φ_i is a bijection from the set $\{A | A$ is an atom in $\mathscr{K}_i$ contained in $\bigcup_{t \in K_i}\{x \in V | x \geqslant t\}\}$ to K_i. In particular, every such atom A is of the form $A = \{x \in V | x \geqslant t\} \backslash \bigcup_{u \in K_i, u \not\leqslant t}\{x \in V | x \geqslant u\}$, where $t = \rho_i(A)$.*

Proof. Let A be an atom in $\mathscr{K}_i$; then there exists a subset $G \subseteq K_i = \{t_1, ..., t_m\}$ such that

$$\mathbf{1}_A = \prod_{s \in G} \mathbf{1}_{\{x \in V | x \geqslant s\}} \cdot \prod_{s \notin G} \mathbf{1}_{\{x \in V | x \geqslant s\}^c}.$$

For $A \subseteq \bigcup_{s \in K_i} \{x \in V | x \geqslant s\}$ we have

$$\mathbf{1}_A \cdot \mathbf{1}_{\{x \in V | x \geqslant s\}^c} = \mathbf{1}_A \cdot (\mathbf{1}_{\bigcup_{j=1}^m \{x \in V | x \geqslant t_j\}} - \mathbf{1}_{\{x \in V | x \geqslant s\}})$$

and

$$\mathbf{1}_{\bigcup_{j=1}^m \{x \in V | x \geqslant t_j\}} = \sum_{j=1}^m \mathbf{1}_{\{x \in V | x \geqslant t_j\}} - \sum_{1 \leqslant j < k \leqslant m} \mathbf{1}_{\{x \in V | x \geqslant t_j, t_k\}}$$
$$+ \cdots + (-1)^m \mathbf{1}_{\{x \in V | x \geqslant t_1, ..., t_m\}}$$

by the sieve formula (A.4). With the $\bigcap$-compatibility of K_i it then follows that $\mathbf{1}_A = \sum_{s \in K_i} c(A, s) \mathbf{1}_{\{x \in V | x \geqslant s\}}$ for some integers $c(A, s)$. Let s_A be a minimal element in $\{s \in K_i | c(A, s) \neq 0\}$; due to $\mathbf{1}_A(s_A) = \sum_{s \in K_i} c(A, s) \mathbf{1}_{\{x \in V | x \geqslant s\}}(s_A) = c(A, s_A) \neq 0$, s_A belongs to the set A. A being an atom in $\mathscr{K}_i$, it follows that $A \subseteq \{x \in V | x \geqslant s_A\}$, and therefore $\varphi_i(A) = s_A$. The rest of the lemma then follows. Q.E.D.

So if $V = \bigcup_{s \in K_i}\{x \in V | x \geqslant s\}$, we get a bijection φ_i from $\{A | A$ an atom in $\mathscr{K}_i\}$ to K_i; if $V \neq \bigcup_{s \in K_i} \{x \in V | x \geqslant s\}$, we introduce an artificial element 0, defined to be smaller than all elements in V, and by setting $\varphi_i(V \backslash \bigcup_{s \in K_i}\{x \in V | x \geqslant s\}) := 0$, we again get a bijection φ_i, but now from $\{A | A$ an atom of $\mathscr{K}_i\}$ to $K_i^0 := K_i \cup \{0\}$.

The existence of such a bijection φ_i is a consequence of the more combinatorial notion of the $\bigcap$-compatibility of K_i. But in fact the $\bigcap$-compatibility of K_i could also be deduced from the existence of φ_i, so that the two conditions are equivalent.

An important consequence is that every decreasing sequence $(A_i)_{i \in \mathbb{N}} \in J(\mathbf{K})$ of atoms $A_i \in \mathscr{K}_i$ can be represented by an increasing sequence $(s_i)_{i \in \mathbb{N}} =: \Phi((A_i)_{i \in \mathbb{N}})$ of elements $s_i := \varphi_i(A_i) \in K_i \cup \{0\}$, with the property that if $t \in K_i \cup \{0\}$, $t \not\leqslant s_i$, then $t \not\leqslant s_{i+1}$. So if

$V = \bigcup_{s \in K_1} \{x \in V | x \geqslant s\}$, let us introduce the measurable space $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ by defining

$$S(K) := \{(s_i)_{i \in \mathbb{N}} | s_i \leqslant s_{i+1}, s_i \in K_i \ \forall i \in \mathbb{N}; t \in K_i, t \nleqslant s_i \Rightarrow t \nleqslant s_{i+1} \ \forall i \in \mathbb{N}\};$$

$$\mathscr{S}(K) := \sigma\{\{(s_i)_{i \in \mathbb{N}} \in S(\mathbf{K}) | s_n = t\} | t \in K_n, n \in \mathbb{N}\}.$$

If $V \neq \bigcup_{s \in K_1} \{x \in V | x \geqslant s\}$, define $(S(\mathbf{K}^0), \mathscr{S}(\mathbf{K}^0))$ as above, but with $(K_i)_{i \in \mathbb{N}}$ replaced by $(K_i^0)_{i \in \mathbb{N}} := (K_i \cup \{0\})_{i \in \mathbb{N}}$, and set

$$S(\mathbf{K}) := S(\mathbf{K}^0) \backslash \nu(0), \qquad \text{where } \nu(0) := (0, 0, 0, \ldots);$$

$$\mathscr{S}(\mathbf{K}) := \mathscr{S}(\mathbf{K}^0)|_{S(\mathbf{K})}.$$

Then the following characterization of $(J(\mathbf{K}), \mathscr{T}(\mathbf{K}))$ is immediate:

LEMMA 1.6. *$(J(\mathbf{K}), \mathscr{T}(\mathbf{K})$ and $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ are isomorphic measurable spaces.*

DEFINITION 1.7. The partial order $\leqslant_S$ defined on $S(K)$ by $(s_i)_{i \in \mathbb{N}} \leqslant_S (t_i)_{i \in \mathbb{N}}$ iff $s_i \leqslant t_i \forall i \in \mathbb{N}$ is called the natural order on $S(\mathbf{K})$.

The partial order $\leqslant_J$ on $J(\mathbf{K})$ is then of course the partial order induced from $(S(\mathbf{K}), \leqslant_S)$ by the isomorphism Φ defined above. Since in this way the spaces $(J(\mathbf{K}), \mathscr{T}(\mathbf{K}), \leqslant_J)$ and $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}), \leqslant_S)$ become equivalent for our purposes, we will mostly be working with $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}), \leqslant_S)$, calling it the inverse limit space associated with $\mathbf{K}$ on V as well.

DEFINITION 1.8. For $s \in V$ let $(A_i)_{i \in \mathbb{N}}$ be the sequence of atoms $A_i \in \mathscr{K}_i$ with $s \in A_i$; $(A_i)_{i \in \mathbb{N}} \in J(K)$. The map $\nu: V \to S(\mathbf{K})$ defined by $\nu(s) := \Phi((A_i)_{i \in \mathbb{N}})$ is called the natural embedding of V into $S(\mathbf{K})$.

LEMMA 1.9. *$\nu: V \to S(\mathbf{K})$ is injective and order-preserving. Furthermore, if $(s_i)_{i \in \mathbb{N}} \in S(K)$, then $(s_i)_{i \in \mathbb{N}} = \bigvee_{j \in \mathbb{N}}^S \nu(s_j)$.*

Proof. For $s \in K_n$, $\nu(s) = (s_i)_{i \in \mathbb{N}}$ with $s_i = s \ \forall i \geqslant n$, so ν is injective, $(s_i)_{i \in \mathbb{N}} = \bigvee_{j \in N}^S \nu(s_j)$ and $\nu(s) \geqslant_S \nu(t)$ implies $s \geqslant t$. If $s \geqslant t$, $\nu(s) = (s_i)_{i \in \mathbb{N}}$, $\nu(t) = (t_i)_{i \in \mathbb{N}}$, then $s \geqslant t \geqslant t_n$, so that for the atom A_n of $\mathscr{K}_n$ containing s we get $s_n \in A_n \subseteq \{x \in V | x \geqslant t_n\}$. Q.E.D.

As a consequence of Lemma 1.9, we will not discern between $\geqslant_V$, $\geqslant_J$ and $\geqslant_S$ hereafter.

Proof of Theorem 1.4. It remains to show that $\mathscr{T}(\mathbf{K}) = \sigma\{\{x \in J(K) | x \geqslant$

$s\}|s \in J(\mathbf{K})\}$, or, equivalently, that $\mathscr{S}(\mathbf{K}) = \sigma\{\{x \in S(\mathbf{K})|x \geqslant s\}|s \in S(\mathbf{K})\}$. Assume $V = \bigcup_{s \in K_1} \{x \in V|x \geqslant s\}$;

$$\begin{aligned}\mathscr{S}(K) &:= \sigma\{\{(s_i)_{i\in\mathbb{N}} \in S(\mathbf{K})|s_n = t\}|t \in K_n, n \in \mathbb{N}\}\\ &= \sigma\{\{(s_i)_{i\in\mathbb{N}} \in S(\mathbf{K})|s_n \geqslant t\}|t \in K_n, n \in \mathbb{N}\}\\ &= \sigma\{\{(s_i)_{i\in\mathbb{N}} \in S(\mathbf{K})|(s_i)_{i\in\mathbb{N}} \geqslant v(t)\}|t \in K_n, n \in \mathbb{N}\}\end{aligned}$$

since for $t \in K_n$, $(s_i)_{i\in\mathbb{N}} \geqslant v(t) \Leftrightarrow s_n \geqslant t$. This proves $\mathscr{S}(K) \subseteq \sigma\{\{x \in S(\mathbf{K})|x \geqslant s\}|s \in S(\mathbf{K})\}$, and the converse inequality holds due to $S(\mathbf{K}) \ni (s_i)_{i\in\mathbb{N}} = \bigvee^S_{j\in\mathbb{N}} v(s_j)$. The case $V \neq \bigcup_{s\in K_1} \{x \in V|x \geqslant s\}$ now follows as well, since then $\mathscr{S}(\mathbf{K}) = \mathscr{S}(\mathbf{K}^0)|_{S(\mathbf{K}^0)\backslash v(0)}$. Q.E.D.

Examples

In Example 1, if $V = D^1$, then $J(\mathbf{K})$ consists of the decreasing sequences $(A_i)_{i\in\mathbb{N}}$ of intervals $A_i = [k2^{-i}, (k+1)\,2^{-i})$, $0 \leqslant k < i2^i$, or $A_i = [i, \infty)$. Such a sequence either converges to a unique element of $[0, \infty)$, or else $A_i =: [a_i, b_i) \downarrow \varnothing$, which can only happen if $b_i = k2^{-j}$, $\forall i \geqslant j$, and $a_i \uparrow k2^{-j}$, or if $A_i = [i, \infty)$, $\forall i \in \mathbb{N}$. Therefore $S(K)$ is equivalent to $[0, \infty] \cup^d D^1\backslash\{0\}$, where D^1 is an extra copy of the dyadic numbers. Similarly, if $V = D^n$, $S(\mathbf{K})$ is equivalent to $[0, \infty]^n \bigcup^d \bigcup^d_{I\subseteq\{1,\ldots,n\}} \{(x_1,\ldots, x_n) \in (0, \infty]^n|x_i \in D^1 \forall i \in I\}$ because for every point $x \in [0, \infty]^n$ which has k sequences $(s_i)_{i\in N}$ in $S(\mathbf{K})$ with $s_i \uparrow x$, we have to add $k-1$ extra copies of x to $[0, \infty]^n$. In Example 2, $S(\mathbf{K})$ is equivalent to $\{U|U$ an open set in $T\} \bigcup^d \mathscr{U}^1$, where $\mathscr{U}^1$ contains extra copies of those open sets $U \subseteq T$ which are the limit of more than one sequence $(U_i)_{i\in N} \in S(\mathbf{K})$. ∎

The inverse limit space of examples 3 and 4 are described by the following general theorem:

THEOREM 1.10. *$(S(\mathbf{K}), \leqslant)$ is a complete poset in the sense that every $\uparrow$-directed set has a lowest upper bound; in particular, if V is $\uparrow$-directed, then $(S(\mathbf{K}), \leqslant)$ has a largest element, denoted by* 1; *if V is a $\bigvee$-semilattice with a smallest element* 0, *then $S(\mathbf{K})$ is a complete lattice (in the standard sense).*

Proof. Let $(s^\iota)_{\iota\in I}$ be a $\uparrow$-directed set in $S(\mathbf{K})$, $s^\iota = (s^\iota_i)_{i\in\mathbb{N}}$. Then $(s^\iota_n)_{\iota\in I}$ is directed in the finite set K_n, so $\exists\tau \in I$ with $\bigvee_{\iota\in I} s^\iota_n = s^\tau_n =: t_n \in K_n$. But then $s^\iota_i \leqslant s^\tau_i \;\forall i \leqslant n$, $\forall\iota \in I$, so that $t_i = s^\tau_i \;\forall i \leqslant n$, and therefore $(t_i)_{i\in\mathbb{N}} \in S(K)$, $\bigvee_{\iota\in I} s^\iota = (t_i)_{i\in\mathbb{N}}$. If V is $\uparrow$-directed, $t_n \in V$ with $t_n \geqslant s$, $\forall s \in K_n$, and A_n, is the atom in $\mathscr{K}_n$ containing t_n, then $\varphi_n(A_n) =: s_n \geqslant s$, $\forall s \in K_n$, which proves $(s_n)_{n\in\mathbb{N}}$ to be the largest element in $S(\mathbf{K})$. If V is a $\bigvee$-semilattice, and $(s_i)_{i\in\mathbb{N}}$, $(t_i)_{i\in N} \in S(\mathbf{K})$, $v(s_n \vee t_n) = v(s_n) \vee v(t_n)$ defines an increasing sequence in

$S(\mathbf{K})$ with lowest upper bound $(u_i)_{i\in\mathbb{N}}$. $(u_i)_{i\in\mathbb{N}} = (s_i)_{i\in\mathbb{N}} \vee (t_i)_{i\in\mathbb{N}}$, and if $(s_i)_{i\in\mathbb{N}} = \nu(s)$, $(t_i)_{i\in\mathbb{N}} = \nu(t)$, $u_i \geqslant s_i \vee t_i$ implies $s \vee t = \bigvee_{i\in\mathbb{N}} u_i$; $\nu(s \vee t) \geqslant (u_i)_{i\in\mathbb{N}}$, but equality does not hold generally. Q.E.D.

Note that for every pair $((s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}) \in S(\mathbf{K}) \times S(\mathbf{K})$ there exists a unique index $n \in \mathbb{N} \cup \{0\}$ such that $s_i = t_i$, $\forall i \leqslant n$, and $s_i \neq t_i$, $\forall i > n$, due to the definition of $S(\mathbf{K})$.

THEOREM 1.11. *Assume* $V = \bigcup_{s\in K_1} \{x \in V \mid x \geqslant s\}$;

(i) *If* $\mathscr{E}$ *is the topology on* $S(\mathbf{K})$ *defined by the open sets* $U_s^n := \{(s_i)_{i\in\mathbb{N}} \in S(\mathbf{K}) \mid s_n = s\}$, $s \in K_n$, $n \in \mathbb{N}$, *then* $(S(K), \mathscr{E})$ *is a compact space. The same follows for the topology* $\mathscr{E}_{\geqslant}$ *defined by the closed sets* $\{x \in S(\mathbf{K}) \mid x \geqslant s\}$, $s \in S(\mathbf{K})$.

(ii) *If d is the metric on* $S(\mathbf{K})$ *defined by*

$$d((s_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}) := 2^{-n} \qquad \text{iff } s_n = t_n,\ s_{n+1} \neq t_{n+1},$$

then $\mathscr{E}$ *is the topology induced by d, so that* $(S(\mathbf{K}), d)$ *is a complete metric space with the dense subset* $\nu(V)$.

If $V \neq \bigcup_{s\in K_1} \{x \in V \mid x \geqslant s\}$, *then the above statements are true for* $S(\mathbf{K}^0) = S(K) \cup \{\nu(0)\}$.

Proof. K_n is compact with respect to the discrete topology, and the map $\psi_n : K_{n+1} \to K_n$ defined by $\psi_n(s) := \varphi_n(A_s^n)$, A_s^n the atom in $\mathscr{K}_n$ containing s, is continuous. According to Theorem 2.6 of [19, p. 136], $S(\mathbf{K})$ is then a compact space with respect to the topology induced by the open and closed sets U_s^n. Using $\mathscr{E}_{\geqslant} \subseteq \mathscr{E}$, (i) follows. Part (ii) is then immediate, since $\lim_{i\to\infty} \nu(s_i) = (s_i)_{i\in\mathbb{N}}$ with respect to d if $(s_i)_{i\in\mathbb{N}} \in S(\mathbf{K})$. Q.E.D.

Remark. The dual results of this paragraph (i.e., with $\leqslant$ consequently replaced by $\geqslant$ and vice versa) are of course also true.

2. QUASIMARTINGALES ON EXHAUSTIBLE POSETS

Let V be an exhaustible poset with a fixed regularly exhausting sequence $\mathbf{K} = (K_i)_{i\in\mathbb{N}}$. Since K_i is a finite poset $\forall i \in \mathbb{N}$, we can apply Definition A.1 of the Appendix to get the Möbius function $\mu^i : K_i \times K_i \to \mathbb{Z}$ of K_i, and can then define the upper difference operator D_s^i for K_i at the point $s \in K_i$ by $D_s^i f := \sum_{t\in K_i, t\geqslant s} \mu^i(s,t) f(t)$, where f is any map on K_i with values in an additive group G. For details, see the Appendix, and in particular Lemma A.2.

To illustrate the main idea of this paragraph, let us assume that $V = \bigcup_{s\in K_1} \{x \in V \mid x \geqslant s\}$, and that f is a function of bounded variation, which means that $\sup_{i\in\mathbb{N}} \sum_{s\in K_i} |D_s^i f| < \infty$. If A_s^i denotes the atom in $\mathscr{K}_i$

containing $s \in K_i$, $\mu_i(A_s^i) := D_s^i f$ then defines a sequence $(\mu_i)_{i \in \mathbb{N}}$ of measures on $(V, \mathscr{K}_i)_{i \in \mathbb{N}}$ such that $\mu_i\{x \in V | x \geqslant s\} = f(s)$, $\forall s \in K_i$, $i \in \mathbb{N}$. As we will show, the $(\mu_i)_{i \in \mathbb{N}}$ are consistent, and $\sup_{i \in \mathbb{N}} \|\mu_i\| = \sup_{i \in \mathbb{N}} \sum_{s \in K_i} |D_s^i f| < \infty$, so that they define a finite signed measure μ on $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ associated with f by means of $\mu\{x \in S(\mathbf{K}) | x \geqslant v(s)\} = f(s)$, $\forall s \in V$.

This construction is applicable to far more general situations. First, $V = \bigcup_{s \in K_1} \{x \in V | x \geqslant s\}$ is not necessary; second, f does not have to be a function: if $(\Omega, \mathscr{F}, P)$ is a probability space and $(\mathscr{F}_s)_{s \in V}$ is a family of sub-σ-algebras of $\mathscr{F}$ such that $s \geqslant t$ implies $\mathscr{F}_s \supseteq \mathscr{F}_t$, we can work with an appropriate set $Q = (Q_s)_{s \in V}$ of measures:

Definition 2.1. For $s \in V$ let Q_s be a σ-additive measure on $(\Omega, \mathscr{F})$ with values in a Banach space E. The set $Q = (Q_s)_{s \in V}$ is called an E-quasimartingale on $\mathbf{K}$ if the $\mathbf{K}$-semivariation of Q defined by

$$\mathbf{K} - \mathrm{sv}(Q) := \sup \left\{ \left\| \sum_{s \in K_i} \sum_{F \in \Pi_s} \varepsilon_{s,F} D_s^i Q(F) \right\| \,\middle|\, \Pi_s \text{ is a measurable partition of } (\Omega, \mathscr{F}_s); \varepsilon_{s,F} \in \{-1, +1\}; i \in \mathbb{N} \right\}$$

is finite. Here, a measurable partition of $(\Omega, \mathscr{F}_s)$ is a finite number of disjoint $\mathscr{F}_s$-measurable sets. If, in addition, the $\mathbf{K}$-variation of Q

$$\mathbf{K}\text{-var}(Q) := \sup_{i \in \mathbb{N}} \sum_{s \in K_i} \mathrm{var}(D_s^i Q|_{\mathscr{F}_s})$$

is finite, where $\mathrm{var}(D_s^i Q|_{\mathscr{F}_s})$ denotes the total variation of the measure $D_s^i Q$ on the space $(\Omega, \mathscr{F}_s)$, then Q is called an E-quasimartingale of bounded variation on $\mathbf{K}$.

Examples

In Example 1, if $V = D^1$, $|\Omega| = 1$, the $\mathbb{R}$-quasimartingales on $\mathbf{K}$ are precisely the functions of bounded variation; more generally, if $(X_s)_{s \in D^n}$ is a $(\mathscr{F}_s)_{s \in D^n}$ – adapted quasimartingale in the sense of [9, 10], then $Q := (X_s \cdot P)_{s \in D^n}$ is an $\mathbb{R}$-quasimartingale on $\mathbf{K}$.

In Example 2, if $f\colon V \to E$ is a map with values in a Banach space E,

$$D_A^i f = f(A) - \sum_{B \in K_i, |B - A| = 1} f(B) + \sum_{B \in K_i, |B - A| = 2} f(B) - \cdots \pm f(V),$$

where $|B-A|=m$ means $A \subseteq B$ and there exist precisely m sets $A \neq C \in K_i$ with $A \subseteq C \subseteq B$. In Example 4,

$$D_s^i f = f(s) - \sum_{t \in K_i, t \gg s} f(t)$$

where $t \gg s$ means $t > s$ such that $t \geqslant u > s$ for $u \in K_i$ implies $u = t$. ∎

As in [8, 9], the measure associated with an E-quasimartingale on $\mathbf{K}$ will be defined on the predictable sets $\mathscr{P}(\mathbf{K})$ of the space $S(\mathbf{K}) \times \Omega$:

DEFINITION 2.2. $\mathscr{P}(\mathbf{K}) := \sigma\{\{x \in S(\mathbf{K}) | \geqslant \nu(s)\} \times F_s | s \in V, F_s \in \mathscr{F}_s\}$ is called the σ-algebra of the predictable sets on $S(\mathbf{K}) \times \Omega$.

Since our construction relies on Theorem 1.3, we require some regularity conditions on the underlying σ-algebras:

ASSUMPTION 2.3. $(\mathscr{F}_s)_{s \in V}$ is an isotone sequence of sub-σ-algebras of $\mathscr{F}$ such that

(i) $(\Omega, \mathscr{F}_s)$ is a standard Borel space $\forall s \in V$;

(ii) if $(s_i)_{i \in N} \in S(\mathbf{K})$, then there does not exist a decreasing sequence of atoms B_i in $\mathscr{F}_{s_i}$ with $\bigcap_{i \in \mathbb{N}} B_i = \varnothing$.

We could do without (ii), but would then have to construct the measure on some larger inverse limit space than $S(\mathbf{K}) \times \Omega$.

Remark. The definition of $\mathscr{P}(\mathbf{K})$ needs some justification since in the case $V = \mathbb{Q}_+^n$ the predictable sets are normally defined in a different way by setting $\mathscr{P}(\mathbf{K}) := \sigma\{(a, b] \times F_a | F_a \in \mathscr{F}_a; a, b \in \mathbb{Q}_+^n\}$, where $(a, b] := \{x \in \mathbb{Q}_+^n | \ a \ll x \leqslant b\}$ and $a = (a_1, \ldots, a_n) \ll (b_1, \ldots, b_n) = b$ iff $a_i < b_i$, $i = 1, \ldots, n$.

By analogy we would define $\mathscr{P}(\mathbf{K})$ as the σ-algebra generated by the sets $B_s^i \times F$, where B_s^i is of the form $\{x \leqslant s\} \backslash \bigcup_{t \in K_i, t \ngeq s} \{x \leqslant t\}$ and F belongs to $\mathscr{F}_u$ for an appropriate $u \in V$ with $\nu(u) \ll x$, $\forall x \in B_s^i$. Unfortunately a general definition for the strong partial ordering $\ll$ is not apparent. For example, even the most simple assumption that $s \ll t$ should imply $s \leqslant t$ leads to unsatisfactory situations: take the basic Example 2; then, if every set B_s^i contains at least two disjoint open sets, $\inf B_s^i = \varnothing$ and $\mathscr{P}(\mathbf{K})$ degenerates to $\mathscr{S}(\mathbf{K}) \times \mathscr{F}_\varnothing$.

We can avoid these problems if we note that for $V = \mathbb{Q}_+^n$ and an isotone rightcontinuous sequence $(\mathscr{H}_s)_{s \in V}$ of σ-algebras,

$$\mathscr{P}(\mathbf{K}) = \{[s, \infty) \times F | F \in \mathscr{H}_{s-}, s \in \mathbb{Q}_+^n\}, \quad \text{where} \quad \mathscr{H}_{s-} := \sigma\left(\bigcup_{t \ll s, t \in \mathbb{Q}_+^n} \mathscr{H}_t\right);$$

if we set $\mathscr{F}_s := \mathscr{H}_{s-}$, this is roughly Definition 2.2, and the problem of the strong partial order $\preccurlyeq$ is reduced to the simpler question of working with the appropriate σ-algebras $(\mathscr{F}_s)_{s\in V}$. For the case $V = \mathbb{R}^n_+$, we will show the equivalence between the usual definition and our approach in more detail in Section 5. ∎

Let $Q = (Q_s)_{s\in V}$ be a set of σ-additive measures Q_s on $(\Omega, \mathscr{F}_s)$ with values in a Banach space E.

THEOREM 2.4. *Q is an E-quasimartingale on* $\mathbf{K}$ *iff there exists a σ-additive measure P^Q, defined on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$, with values in the bidual E'' of E, such that*

(i) *the semivariation* $\mathrm{sv}(P^Q) := \sup\{\|h \circ P^Q\| \mid h \in E'$ *the dual of* E, $\|h\| \leqslant 1\}$ *of P^Q is finite;*

(ii) $P^Q[\{x \in S(\mathbf{K}) \mid x \geqslant v(s)\} \times F_s] = \Theta \circ Q_s(F_s)\ \forall s \in V, \qquad \forall F_s \in \mathscr{F}_s$, *where $\Theta: E \to E''$ is the canonical embedding of E into E''.*

P^Q is the unique E''-valued σ-additive measure on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ satisfying (ii) *for a given E-quasimartingale Q on* $\mathbf{K}$. *In addition,* $\mathrm{sv}(P^Q) = \mathbf{K}\text{-}\mathrm{sv}(Q)$.

We call P^Q the measure associated with the E-quasimartingale Q.

Proof. Assume that Q is an E-quasimartingale on $\mathbf{K}$ and that $V = \bigcup_{s\in K_1}\{x \geqslant s\}$; for $t \in K_n$ let A^n_t be the atom of $\mathscr{H}_n$ containing t and set ${}^S A^n_t = \{(t_i)_{i\in\mathbb{N}} \in S(\mathbf{K}) \mid t_n = t\}$. Define the measure P^Q_1 on the σ-algebra $\mathscr{P}_n := \{\bigcup_{t\in K_n} {}^S A^n_t \times F_t \mid F_t \in \mathscr{F}_t\}$ by

$$P^Q_1\left[\bigcup_{t\in K_n}^{d} {}^S A^n_t \times F_t\right] = \sum_{t\in K_n} D^n_t Q(F_t).$$

This definition of P^Q_1 on $(\mathscr{P}_i)_{i\in\mathbb{N}}$ is consistent since for ${}^S A^n_t = \bigcup^d_{j=1,\ldots,m} {}^S A^{n+1}_{t_j}$ we have by Lemma A.3 $\forall F_t \in \mathscr{F}_t$

$$\begin{aligned} P^Q_1\left[\bigcup_{j=1,\ldots,m}^{d} {}^S A^{n+1}_{t_j} \times F_t\right] &= \sum_{j=1}^{m} D^{n+1}_{t_j} Q(F_t) \\ &= D^n_t\, Q(F_t) \\ &= P^Q_1[{}^S A^n_t \times F_t]. \end{aligned}$$

Therefore, for all h in the dual E' of E with $\|h\| \leqslant 1$, $(h \circ P^Q_1|_{\mathscr{P}_i})_{i\in\mathbb{N}}$ is a consistent sequence of signed measures with $\sup\{\|h \circ P^Q_1|_{\mathscr{P}_i}\| \mid i \in \mathbb{N}\} \leqslant \sup\{\mathrm{sv}(P^Q_1|_{\mathscr{P}_i}) \mid i \in \mathbb{N}\} = \mathbf{K} - \mathrm{sv}(Q) < \infty$. If $({}^S A^i_{t_i} \times F_{t_i})_{i\in\mathbb{N}}$ is a decreasing sequence of atoms of $\mathscr{P}_i$ with $t_i \in K_i$, then $\bigcap_{i\in\mathbb{N}} ({}^S A^i_{t_i} \times F_{t_i}) \supseteq \bigcap_{i\in N}({}^S A^i_{t_i}) \cap \bigcap_{i\in N}(F_{t_i})$, where $(t_i)_{i\in N} \in \bigcap_{i\in N} {}^S A^i_{t_i}$ and $\bigcap_{i\in N} F_{t_i} \neq \varnothing$ according to our Assumption 2.3. Since $\mathscr{P}_i$ is σ-isomorphic to the direct product

$\prod_{t\in K_i}\mathscr{F}_t$, $\mathscr{P}_i$ is standard Borel by Theorem 2.3 of [19, p.135], so that we can apply Theorem 1.1 to get a unique measure P_h on $(S(\mathbf{K})\times\Omega, \sigma(\bigcup_{i\in\mathbb{N}}\mathscr{P}_i))$ with $P_h = h\circ P_1^Q$ on $\bigcup_{i\in\mathbb{N}}\mathscr{P}_i$. This proves the weak-$\sigma$-additivity of P_1^Q on $\bigcup_{i\in\mathbb{N}}\mathscr{P}_i$; because $\mathrm{sv}(P_1^Q|_{\bigcup_{i\in\mathbb{N}}\mathscr{P}_i}) = \sup_{i\in\mathbb{N}}\mathrm{sv}(P_1^Q|_{\mathscr{P}_i}) = \mathbf{K}-\mathrm{sv}(Q)<\infty$, P_1^Q then has a unique norm-σ-additive extension P^Q to $\sigma(\bigcup_{i\in\mathbb{N}}\mathscr{P}_i)=\mathscr{P}(\mathbf{K})$ with $\mathrm{sv}(P^Q)=\mathbf{K}-\mathrm{sv}(Q)$ by Theorem 10.6 of [15]. For $t\in K_i, F_t\in\mathscr{F}_t$ we have

$$\begin{aligned} P^Q[\{x\in S(\mathbf{K})|x\geqslant v(t)\}\times F_t] &= P^Q\left[\overset{d}{\bigcup_{u\in K_i, u\geqslant t}}\{(t_j)_{j\in\mathbb{N}}\in S(K)|t_i=u\}\times F_t\right] \\ &= \sum_{u\in K_i, u\geqslant t} D_u^i Q(F_t) \\ &= Q_t(F_t), \end{aligned}$$

and with $V=\bigcup_{i\in\mathbb{N}}K_i$ this proves (ii).

Now let us assume that $V\neq\bigcup_{s\in K_1}\{x\geqslant s\}$. As usual, we introduce the artificial smallest element 0 and define $V^0 := V\cup\{0\}$ and $\mathbf{K}^0 = (K_i^0)_{i\in\mathbb{N}} := (K_i\cup\{0\})_{i\in\mathbb{N}}$, with associated inverse limit space $(S(\mathbf{K}^0), \mathscr{S}(\mathbf{K}^0))$ and predictable sets $\mathscr{P}(\mathbf{K}^0)$; set $\mathscr{F}_0=\{\varnothing,\Omega\}$ and $Q_0(\Omega)=0$. Since for the upper difference operator $D_t^{i,0}$ on K_i^0 we have $D_t^{i,0}=D_t^i\,\forall t\in K_i$ and $D_0^{i,0}Q = Q_0-\sum_{t\in K_i}D_t^{i,0}Q$, $\forall i\in\mathbb{N}$, $\mathbf{K}^0-\mathrm{sv}(Q)\leqslant 2\mathbf{K}-\mathrm{sv}(Q)<\infty$ so that the measure $P^{Q,0}$ associated with $(Q_t)_{t\in V^0}$ exists on $(S(\mathbf{K}^0)\times\Omega, \mathscr{P}(\mathbf{K}^0))$. Then, with

$$\bigcup_{i\in\mathbb{N}}\bigcup_{s\in K_i}\{x\in S(\mathbf{K}^0)|x\geqslant v^0(s)\} = S(\mathbf{K}^0)\backslash\{v^0(0)\} = S(\mathbf{K}),$$

it easily follows that $P^Q := P^{Q,0}|_{S(\mathbf{K})\times\Omega}$ has the stated properties.

Inversely, if a measure P^Q exists with the stated properties, then $\mathrm{sv}(P^Q)=\mathbf{K}-\mathrm{sv}(Q)$ immediately proves Q to be an E-quasimartingale on $\mathbf{K}$.

Q.E.D.

COROLLARY 2.5. *Let Q be an E-quasimartingale on* $\mathbf{K}$. *If $(t_i)_{i\in\mathbb{N}}$ is a monotone (increasing or decreasing) sequence in V, there exists a unique E''-valued measure Q_∞ on $\sigma(\lim_{i\to\infty}\mathscr{F}_{t_i})$ with* $\mathrm{sv}(Q_\infty)<\infty$, *such that*

$$\underset{i\to\infty}{\text{norm-lim}}\;\Theta\circ Q_{t_i}(F) = Q_\infty(F),$$

$\forall F\in\bigcup_{i\in\mathbb{N}}\mathscr{F}_{t_i}$ *if $(t_i)_{i\in\mathbb{N}}$ is increasing,* $\forall F\in\bigcap_{i\in\mathbb{N}}\mathscr{F}_{t_i}$ *if $(t_i)_{i\in\mathbb{N}}$ is decreasing.*

Proof. If P^Q is the measure associated with Q, then define

$$Q_\infty(F) := P^Q\left[\left(\lim_{i\to\infty}\{x \in S(\mathbf{K}) | x \geqslant \nu(t_i)\}\right) \times F\right],$$

$$\forall F \in \sigma\left(\lim_{i\to\infty} \mathscr{F}_{t_i}\right). \quad \text{Q.E.D.}$$

If in particular $(t_i)_{i\in\mathbb{N}} = t \in S(\mathbf{K})$, we define $Q_t = \lim_{i\to\infty} \Theta \circ Q_{t_i}$ in the sense of Corollary 2.5, and $\mathscr{F}_t := \sigma(\bigcup_{i\in\mathbb{N}} \mathscr{F}_{t_i})$. Since for $s \in V$ $Q_{\nu(s)} = \Theta \circ Q_s$ and $\mathscr{F}_{\nu(s)} = \mathscr{F}_s$, this notation is consistent.

COROLLARY 2.6. *Let Q be an E-quasimartingale on* $\mathbf{K}$ *with associated measure P^Q. Then, $\forall t \in S(\mathbf{K})$, $\forall F_t \in \mathscr{F}_t$,*

$$P^Q[\{x \in S(\mathbf{K}) | x \geqslant t\} \times F_t] = Q_t(F_t).$$

If Q is an E-quasimartingale of bounded variation on $\mathbf{K}$, the last three results can be strengthened:

THEOREM 2.7. *Q is an E-quasimartingale of bounded variation on* $\mathbf{K}$ *iff there exists a measure P^Q as in Theorem* 2.4, *but with values in E and with finite total variation* $\operatorname{var}(P^Q)$. *Furthermore, $\operatorname{var}(P^Q) = \mathbf{K}\text{-}\operatorname{var}(Q)$.*

Proof. Just as in Theorem 2.4. However, note that $\operatorname{var}(P_1^Q) = \sup_{i\in\mathbb{N}} \sum_{s\in K_i} \operatorname{var}(\dot{D}_s^i Q|_{\mathscr{F}_s}) = \mathbf{K}\text{-}\operatorname{var}(Q)$, so that the Caratheodory–Hahn–Kluvanek extension theorem is applicable if $\mathbf{K}\text{-}\operatorname{var}(Q) < \infty$. Q.E.D.

As a consequence, if $\mathbf{K}\text{-}\operatorname{var}(Q) < \infty$, Corollaries 2.5 and 2.6 remain true if the embedding Θ is omitted, and the measures Q_∞ and Q_t, $t \in S(\mathbf{K})$, assume values in the space E.

The following two special cases, although immediate, are of particular interest:

DEFINITION 2.8. A map $f: V \to E$, E a Banach space, is of bounded semivariation on $\mathbf{K}$ if $\sup\{\|\sum_{s\in K_i} \varepsilon_s D_s^i f\| \,|\, |\varepsilon_s| \leqslant 1, i \in \mathbb{N}\} =: \mathbf{K} - \operatorname{sv}(f) < \infty$, and of bounded variation on $\mathbf{K}$ if $\sup\{\sum_{s\in K_i} \|D_s^i f\| \,|\, i \in \mathbb{N}\} =: \mathbf{K} - \operatorname{var}(f) < \infty$.

COROLLARY 2.9. *A map $f: V \to E$ is of bounded semivariation on K iff there exists an E''-valued σ-additive measure P^f on $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ with $\operatorname{sv}(P^f) < \infty$ and with $P^f[\{x \in S(\mathbf{K}) | x \geqslant \nu(s)\}] = \Theta \circ f(s)$, $\forall s \in V$; f is of bounded variation on K iff P^f can be chosen to be E-valued with $\operatorname{var}(P^f) < \infty$.*

Definition 2.10. A real-valued $(\mathscr{F}_s)_{s\in V}$-adapted L^1-process $X=(X_s)_{s\in V}$ is a quasimartingale on $\mathbf{K}$ if $\mathbf{K}\text{-var}(X) := \sup\{\sum_{s\in K_i} \|E[D_s^i X|\mathscr{F}_s]\|_{L^1} | i\in \mathbb{N}\} < \infty$.

Corollary 2.11. *An adapted L^1-process X is a quasimartingale on $\mathbf{K}$ iff there exists a signed measure P^X on $(S(\mathbf{K})\times\Omega, \mathscr{P}(\mathbf{K}))$ with $\|P^X\|<\infty$ and $P^X[\{x\in S(\mathbf{K})| x\geqslant \nu(s)\}\times F_s] = E[X_s; F_s]$, $\forall s\in V$, $\forall F_s\in\mathscr{F}_s$.*

This follows from Theorem 2.4 applied to the $\mathbb{R}$-quasimartingale on $\mathbf{K}$ defined by $Q_s := X_s\cdot P$, $s\in V$.

Lemma 2.12. *Let X be a quasimartingale on $\mathbf{K}$ with associated measure P^X. If $s=(s_i)_{i\in\mathbb{N}}\in S(\mathbf{K})$ and $F_s\in\mathscr{F}_s$, then*

$$\begin{aligned} P^X[\{x\in S(\mathbf{K})|x\geqslant s\}\times F_s] &= E[X_s; F_s] && \text{if } s\in \nu(V)\\ &= E[X_s;F_s]+\mu_s(F_s) && \text{if } s\notin \nu(V), \end{aligned}$$

where $X_s :=$a.s.-$\lim_{i\to\infty} X_{s_i}$ and μ_s is a finite signed measure on $\mathscr{F}_s$ orthogonal to $P|_{\mathscr{F}_s}$.

Proof. Apply Corollary 2.5 to the sequence $(X_{s_i}\cdot P)_{i\in\mathbb{N}}$; if $s\notin\nu(V)$, $(X_{s_i})_{i\in\mathbb{N}}$ is a process of bounded variation in the classical sense, i.e., $\sum_{i\in\mathbb{N}} E[|E[X_{s_{i+1}}-X_{s_i}|\mathscr{F}_{s_i}]|]<\infty$. Therefore, X_s is defined, and there exists the Doob decomposition $X=M+A$ into a martingale $M=(M_{s_i})_{i\in\mathbb{N}}$ and a predictable process $A=(A_{s_i})_{i\in\mathbb{N}}$ with $A_s := L^1-\lim_{i\to\infty} A_{s_i}$. The martingale M defines a consistent sequence of measures $(M_{s_i}\cdot P)_{i\in\mathbb{N}}$ on the standard Borel spaces $(\Omega, \mathscr{F}_{s_i})_{i\in\mathbb{N}}$, and therefore induces a measure μ^M on $(\Omega,\mathscr{F}_s)$ by Theorem 1.1. According to classical results, $\mu^M = M_s\cdot P+\mu_0^M$ with

$$M_s = \text{a.s.-}\lim_{i\to\infty} M_{s_i},\ \mu_0^M\left(\left\{\sup_{i\in\mathbb{N}}|M_{s_i}|=\infty\right\}^c\right)=0 \text{ and}$$

$$P\left\{\sup_{i\in\mathbb{N}}|M_{s_i}|=\infty\right\}=0. \quad \text{Q.E.D.}$$

3. Decomposition of Quasimartingales

Let V be an exhaustible poset and let $(\mathscr{F}_s)_{s\in V}$ satisfy Assumption 2.3. Q is a set of measures as usual, X an adapted L^1-process and f a map from V to E.

Definition 3.1. $\mathbb{R}$-quasimartingales Q and X on $\mathbf{K}$ are called S-

processes on $\mathbf{K}$ if $D_s^i Q|_{\mathscr{F}_s} \geqslant 0$ and $E[D_s^i X|\mathscr{F}_s] \geqslant 0$, respectively, $\forall s \in K_i$, $i \in \mathbb{N}$. A function f of bounded variation on $\mathbf{K}$ is a distribution function on $\mathbf{K}$ if $D_s^i f \geqslant 0$, $\forall s \in K_i, i \in \mathbb{N}$.

The term S-process has been taken from Cairoli [4].

LEMMA 3.2. *Q and X are S-processes on* $\mathbf{K}$ *iff P^Q and P^X exist and are finite positive measures. f is a distribution function iff P^f exists and is a finite positive measure.*

In particular, an S-process is a positive supermartingale, but the converse is not true, just as a distribution function is positive and monotonically decreasing, without the converse being correct.

Together with the results of Sections 4 and 5, the following theorem will be seen to be a generalization of the Rao decomposition of a quasimartingale on $\mathbb{R}_+$:

THEOREM 3.3. *Q is an $\mathbb{R}$-quasimartingale on* $\mathbf{K}$ *iff there exist two S-processes Q^+ and Q^- such that $Q = Q^+ - Q^-$. The decomposition can be chosen in such a way that* $\mathbf{K}\text{-var}(Q) = \mathbf{K}\text{-var}(Q^+) + \mathbf{K}\text{-var}(Q^-)$, *and then it is unique. The analogous statements hold for X and f.*

Proof. Let $P^Q = P^+ - P^-$ be the Hahn decomposition of P^Q and Q^+, Q^- the S-processes associated with P^+, P^-. The same argument holds for a function f and for an $\mathbb{R}$-quasimartingale X on $\mathbf{K}$, for the latter because $|P^X|[\{x \in S(\mathbf{K}) | x \geqslant v(s)\} \times \cdot\,] \ll P|_{\mathscr{F}_s}$, $\forall s \in V$. Q.E.D.

DEFINITION 3.4. Q is an E-martingale if Q_s restricted to $\mathscr{F}_t$ is equal to $Q_t \forall s \geqslant t \in V$. X is a martingale if $E[X_s|\mathscr{F}_t] = X_t$, $\forall s \geqslant t \in V$.

X is a martingale iff $(Q_s)_{s \in V} := (X_s \cdot P)_{s \in V}$ is an $\mathbb{R}$-martingale.

LEMMA 3.5. *Let $S(\mathbf{K})$ have a largest element 1. Then the following statements are equivalent*:

(i) *Q is an E-martingale with* $\mathbf{K}\text{-sv}(Q) < \infty$;

(ii) *Q is an E-martingale with* $\sup\{\text{sv}(Q_s) | s \in V\} < \infty$;

(iii) *P^Q with* $\text{sv}(P^Q) < \infty$ *exists and is concentrated on* $\{1\} \times \Omega$.

In particular, Lemma 3.5 combined with Theorem 3.3 gives the Krickeberg decomposition for a martingale X.

Proof. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are clear, so let Q be an E-martingale with $\sup\{\text{sv}(Q_s) | s \in V\} < \infty$. For a fixed $n \in \mathbb{N}$ let μ be the Möbiusfunction on K_n (see Definition A.1). Since $1 = (s_i)_{i \in \mathbb{N}}$ is the largest element in $S(\mathbf{K})$, s_n is the largest element in K_n; for $s \in K_n$ and $F \in \mathscr{F}_s$ this implies

$$D_s^n Q(F) := \sum_{u \in K_n, u \geqslant s} \mu(s, u)\, Q_u(F)$$
$$= Q_s(F) \left\{ \mu(s, s_n) + \sum_{u \in K_n, s \leqslant u < s_n} \mu(s, u) \right\}$$
$$= \begin{cases} Q_{s_n}(F) & \text{if} \quad s = s_n \\ 0 & \text{if} \quad s \neq s_n \end{cases}$$

because $\mu(s, s_n) := -\sum_{r \in K_n, s \leqslant r < s_n} \mu(s, r)$. Therefore $\mathbf{K}\text{-sv}(Q) = \sup\{\text{sv}(Q_{s_n}) | n \in \mathbb{N}\} = \sup\{\text{sv}(Q_s) | s \in V\} < \infty$, and since $P^Q[\{(t_i)_{i \in \mathbb{N}} \in S(K) | t_n = s\} \times F] = D_s^n Q(F)$, P^Q vanishes on $\{x \in S(\mathbf{K}) | x \geqslant v(s_n)\}^c \times \Omega$. Q.E.D.

For $i \in \mathbb{N}$ and $s \in K_i$ denote the atom in $\mathscr{K}_i$ containing s by A_s^i.

$$J_i := \{s \in K_i | \exists t \in A_s^i,\ t \text{ a maximal element in } V\}$$
$$\cup \{s \in K_i | \exists t = (t_j)_{j \in \mathbb{N}} \in S(\mathbf{K}) \text{ with } t_i = s \text{ and } t \nleqslant v(u)\ \forall u \in V\}.$$

Definition 3.6. $\partial S := \bigcap_{i \in \mathbb{N}} (\bigcup_{s \in J_i} \{(t_j)_{j \in \mathbb{N}} \in S(\mathbf{K}) | t_i = s\})$ is the upper boundary of $S(\mathbf{K})$.

This is not the only plausible definition for the upper boundary, but it has the advantage that ∂S is a measurable set of $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ and that ∂S contains the maximal elements of $v(V)$ as well as those elements of $S(\mathbf{K})$ that are not dominated by any element of $v(V)$. In Example 1, ∂S is equivalent to $\{(s_1, \ldots, s_n) \in S(\mathbf{K}) | \exists i \in \{1, \ldots, n\} \text{ with } s_i = \infty\}$.

Definition 3.7. $\mathbb{R}$-quasimartingales Q and X on $\mathbf{K}$ are called *M-processes* on $\mathbf{K}$ if $\forall i \in \mathbb{N}\ D_s^i Q|_{\mathscr{F}_s} = 0$ and $E[D_s^i X | \mathscr{F}_s] = 0$, respectively, $\forall s \in K_i$ such that $(t_j)_{j \in N} \in \partial S \Rightarrow t_i \neq s$. S-processes Q and X on $\mathbf{K}$ are called potentials on $\mathbf{K}$ if $P^Q[\partial S \times \Omega] = 0$ and $P^X[\partial S \times \Omega] = 0$, respectively.

These definitions are consistent with the standard definitions in the case $V = \mathbb{Q}_+^n$ (or $V = \mathbb{R}_+^n$: see Sections 4 and 5).

Lemma 3.8. (i) *An $\mathbb{R}$-quasimartingale Q on $\mathbf{K}$ is an M-process on $\mathbf{K}$ iff P^Q is concentrated on $\partial S \times \Omega$.*

(ii) *Q is an S-process on $\mathbf{K}$ iff there exists a decomposition $Q = N + Z$ into an M-process N and a potential Z.*

(iii) *Assume $\partial S = \bigcup_{j \in \mathbb{N}} \{x \in S(\mathbf{K}) | x \geqslant s^j\}$ for a set $\{s^j | j \in \mathbb{N}\} \subseteq S(\mathbf{K})$. Then the S-processes Q and X on $\mathbf{K}$ are potentials on $\mathbf{K}$ iff $\lim_{i \to \infty} \|Q_{t_i}\| = 0$*

and $\lim_{i\to\infty} X_{t_i} = 0$ a.s. *and in* L^1 *for any sequence* $(t_i)_{i\in\mathbb{N}} \subseteq V$ *with* $v(t_i) \uparrow s \in \partial S$.

Proof. (i) is the definition of an M-process on $\mathbf{K}$, and (ii) derives from splitting P^Q into its restrictions on $\partial S \times \Omega$ and $(\partial S)^c \times \Omega$. For (iii), let Q be a potential; then $\lim_{i\to\infty} Q_{t_i}(\Omega) = P^Q[\{x \in S(\mathbf{K}) | x \geqslant s\} \times \Omega] = 0$ for $s \in \partial S$. Conversely, if $s^j = (s_i^j)_{i\in\mathbb{N}}$, then $P^Q[\{x \in S(\mathbf{K}) | x \geqslant s^j\} \times \Omega] = \lim_{i\to\infty} Q_{s_i^j}(\Omega) = 0$. Q.E.D.

Example

The following example derives from the theory of random fields (see, e.g., Preston [20]). Let $Z = (Z_s)_{s\in R^n}$ be $\mathscr{F}$-measurable random variables on Ω and define

$$\mathscr{F}_U := \sigma\{Z_s | s \in U\} \text{ for } U \text{ open},$$

$$\mathscr{F}_C := \bigcap_{\epsilon>0} \mathscr{F}_{C^\epsilon} \text{ for } C \text{ closed, } C^\epsilon := \left\{t \in \mathbb{R}^n \,\middle|\, \inf_{s\in C} |t-s| < \varepsilon\right\}.$$

Assume $\mathscr{F} = \sigma(Z_s | s \in \mathbb{R}^n)$; as in Example 2, let V' be a countable basis of $\mathbb{R}^n$ consisting of bounded open sets, partially ordered by inclusion, such that V' is a $\vee$-semilattice, and set $V := \{\bar{U} | U \in V', \bar{U} \text{ the closure of } U\}$. Let P be a probability on $(\Omega, \mathscr{F})$ such that

$$P[F_{\bar{U}} | \mathscr{F}_{U^c}] = P[F_{\bar{U}} | \mathscr{F}_{\partial U}]$$

$\forall \mathscr{F}_{\bar{U}}$-measurable sets $F_{\bar{U}}$, $\forall \bar{U} \in V$, where ∂U is the boundary of U; a probability with this property will be called a Markov random field with respect to $(\mathscr{F}_{\bar{U}})_{\bar{U}\in V}$. If Q is a probability on $(\Omega, \mathscr{F})$ such that $Q|_{\mathscr{F}_{\bar{U}}} \ll P|_{\mathscr{F}_{\bar{U}}} \forall \bar{U} \in V$, define $X_{\bar{U}} := dQ/dP|_{\mathscr{F}_{\bar{U}}}$, $X := (X_{\bar{U}})_{\bar{U}\in V}$. Then Q is a random field with the same specification as P, i.e., with $Q[F | \mathscr{F}_{U^c}] = P[F | \mathscr{F}_{U^c}]$ $\forall F \in \mathscr{F}_{\bar{U}}$, $\bar{U} \in V$, if and only if X is a positive normed martingale such that $X_{\bar{U}}$ is $\mathscr{F}_{\partial U}$-measurable $\forall \bar{U} \in V$. If $(\mathscr{F}_{\bar{U}})_{\bar{U}\in V}$ satisfies Assumption 2.3, then P^X defines Q, and in fact $P^X|_{\{1\}\times\cdot} \equiv Q$, where 1 is the largest element in $S(\mathbf{K})$, so that locally absolutely continuous phase transition is completely described by the probability measures P^X with support $1 \times \Omega$ and with $\mathscr{F}_{\partial U}$-measurable random variables $X_{\bar{U}}$.

More generally, let $\tau: \Omega \to S(\mathbf{K})$ be a random variable with $\{\tau \geqslant \bar{U}\} \in \mathscr{F}_{\partial U}$ $\forall \bar{U} \in V$ and set $\mathscr{F}_{\bar{U}}^\tau := \mathscr{F}_{\bar{U}}|_{\{\tau \geqslant \bar{U}\}}$; note that if we were only working on $V'' := \{A \in \mathbb{R}_+^n | t \in A \Rightarrow [0, t] \subseteq A\}$, τ would define a stopping line [17]. If Q^τ is a measure on $(\Omega, \mathscr{F})$ with $Q^\tau|_{\mathscr{F}_{\bar{U}}^\tau} \ll P|_{\mathscr{F}_{\bar{U}}^\tau} \forall \bar{U} \in V$, define $X_{\bar{U}}^\tau := dQ^\tau/dP|_{\mathscr{F}_{\bar{U}}^\tau}$ ($:= 0$ on $\{\tau \geqslant U\}^c$)), $X^\tau := (X_{\bar{U}}^\tau)_{\bar{U}\in V'}$; then Q^τ satisfies $Q^\tau[F \cap \{\tau \geqslant \bar{U}\} | \mathscr{F}_{U^c}] = P[F \cap \{\tau \geqslant \bar{U}\} | \mathscr{F}_{U^c}]$ $\forall F \in \mathscr{F}_{\bar{U}}$ iff X^τ is a normed S-process with $\|P^{X^\tau}\| = E[X_\phi^\tau] = 1$ such that $X_{\bar{U}}^\tau$ is $\mathscr{F}_{\partial U}$-measurable, $X_{\bar{U}}^\tau = 0$ on

$\{\tau \geqslant \bar{U}\}^c$, $\forall \bar{U} \in V$. If τ assumes only non-maximal values in V, "S-process" can be replaced by "potential."

If we denote $\{B \in S(\mathbf{K}) | B \geqslant A\}$ by $[A \rightarrow]$ and define

$$\begin{aligned}
\mathscr{P}_{\bar{U}} &:= \sigma\{[A\rightarrow] \times F | V \ni A \leqslant \bar{U}, F \in \mathscr{F}_A\}, \\
\mathscr{P}_{U^c} &:= \sigma\{[A\rightarrow] \times F | V \ni A \geqslant \bar{U}, F \in \mathscr{F}_{U^c} \cap \mathscr{F}_A\}, \\
\mathscr{P}_{\partial U} &:= \sigma\{[\bar{U}\rightarrow] \times F | F \in \mathscr{F}_{\partial \bar{U}}\},
\end{aligned}$$

$\forall \bar{U} \in V$, then the fact that P is a Markov random field for $(\mathscr{F}_{\bar{U}})_{\bar{U} \in V}$ translates into P^Y, Y the martingale identical 1, being a Markov random field with respect to $(\mathscr{P}_{\bar{U}})_{\bar{U} \in V}$ since

$$\begin{aligned}
P^Y[[A\rightarrow] \times F \,|\, \mathscr{P}_{U^c}] &= P[F | \mathscr{F}_{U^c}]\, \mathbf{1}_{[\bar{U}\rightarrow]} \\
&= P^Y[[A\rightarrow] \times F \,|\, \mathscr{P}_{\partial U}], \qquad \forall [A\rightarrow] \times F \in \mathscr{P}_{\bar{U}}.
\end{aligned}$$

Similarly, if X^τ is a normed S-process such that $X^\tau_{\bar{U}}$ is $\mathscr{F}_{\partial U}$-measurable, $X^\tau_{\bar{U}} = 0$ on $\{\tau \geqslant \bar{U}\}^c$, $\forall \bar{U} \in V$, then P^{X^τ} is a probability on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ such that

(i) P^{X^τ} is a Markov random field with respect to $(\mathscr{P}_{\bar{U}})_{\bar{U} \in V}$;

(ii) P^{X^τ} has support $\{(A, \omega) | \varnothing \leqslant A \leqslant \tau(\omega)\} =: [\varnothing, \tau]$;

(iii) P^{X^τ} and P^Y restricted to $(\mathscr{P}^\tau_{\bar{U}})_{\bar{U} \in V} := (\sigma\{[A\rightarrow] \times (F \cap \{\tau \geqslant A\}) | V \ni A \leqslant \bar{U}, F \in \mathscr{F}_A\})_{\bar{U} \in V}$ have the same specification, i.e.,

$$\begin{aligned}
P^{X^\tau}[[A\rightarrow] \times F | \mathscr{P}_{U^c}] &= P^{X^\tau}[[A\rightarrow] \times (F \cap \{\tau \geqslant A\}) | \mathscr{P}_{U^c}] \\
&= P^Y[[A\rightarrow] \times (F \cap \{\tau \geqslant A\}) | \mathscr{P}_{U^c}],
\end{aligned}$$

$\forall [A\rightarrow] \times F \in \mathscr{P}_{\bar{U}}$;

(iv) P^{X^τ} has the given conditional densities $(X^\tau_{\bar{U}})_{\bar{U} \in V}$ on $(\mathscr{P}^\tau_{\bar{U}})_{\bar{U} \in V}$ with respect to P^Y, i.e.,

$$\begin{aligned}
P^{X^\tau}([\bar{U}\rightarrow] \times F) &= P^{X^\tau}([\bar{U}\rightarrow] \times (F \cap \{\tau \geqslant \bar{U}\})) \\
&= \int \mathbf{1}_{[\bar{U}\rightarrow] \times F} \cdot X^\tau_{\bar{U}} \, dP^Y \\
&= \int \mathbf{1}_{[\bar{U}\rightarrow] \times F}\, \mathbf{1}_{\{\tau \geqslant \bar{U}\}} \cdot X^\tau_{\bar{U}} \, dP^Y.
\end{aligned}$$

Due to properties (i) and (ii) we call P^{X^τ} a Markov random field with the random life space $[\varnothing, \tau]$. Part (iv) then means that P^{X^τ} is the Markov random field with the given conditional density X^τ (with respect to P^Y) on its life space $[\varnothing, \tau]$.

4. Quasimartingales on Separable Posets

If V is exhaustible and $Q = (Q_s)_{s \in V}$ is an E-quasimartingale on $\mathbf{K}$, Corollary 2.6 shows that Q can be extended in a unique way to a set $(Q_s)_{s \in S(\mathbf{K})}$ such that (omitting Θ in the future)

$$P^Q[\{x \in S(\mathbf{K}) | x \geqslant s\} \times F_s] = Q_s(F_s), \qquad \forall F_s \in \mathscr{F}_s \tag{4.1}$$

not only holds for $s \in \nu(V)$, but for all $s \in S(\mathbf{K})$. Now obviously we could also have constructed this measure P^Q if we had started off with $(Q_s)_{s \in W}$, where W is any set such that $\nu(V) \subseteq W \subseteq S(\mathbf{K})$. In particular, $\nu(V)$ is then dense in W in the sense of convergence of monotonically increasing sequences. So, generally speaking, the set V does not have to be exhaustible, as long as it has an appropriate dense set V^d.

Definition 4.1. A partially ordered, not necessarily countable set V is called a separable poset if there exists an exhaustible poset $V^d \subseteq V$ such that $\forall s \in V$ there exists a sequence $(s_i)_{i \in \mathbb{N}}$ with $s_i \in V^d$, $s_i \leqslant s_{i+1}$ $\forall i \in \mathbb{N}$ and $s = \bigvee_{i \in \mathbb{N}} s_i$ (we shall use the notation $s_i \uparrow s$). A regularly exhausting sequence $\mathbf{K} = (K_i)_{i \in \mathbb{N}}$ of V^d is called a regularly separating sequence of V.

As in Section 1, if there exists an $n \in \mathbb{N}$ with $V^d = \bigcup_{s \in K_n} \{x \in V^d | x \geqslant s\}$ ($\Leftrightarrow V = \bigcup_{s \in K_n} \{x \in V | x \geqslant s\}$) we may assume $n = 1$.

Let V be a separable poset with a fixed exhaustible dense subset V^d and a fixed regularly exhausting sequence $\mathbf{K}$ of V^d, and set $\mathscr{K}_i := \sigma\{\{x \in V | x \geqslant s\} | s \in K_i\}$. Then Lemma 1.5 still holds since its proof does not depend on V being countable, so that there exists a bijection φ_i from $\{A | A$ a atom in $\mathscr{K}_i\}$ to K_i if $V = \bigcup_{s \in K_1} \{x \geqslant s\}$, and to $K_i \cup \{0\}$ otherwise. Consequently Lemma 1.6 follows as well, so that the inverse limit space $(J(\mathbf{K}), \mathscr{T}(\mathbf{K}))$ associated with $(V, \mathscr{K}_i)_{i \in \mathbb{N}}$ is isomorph to the inverse limit space $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ associated with $\mathbf{K}$ on V^d as in Section 1. The space $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ furnished with the natural order $\geqslant$ will therefore also be called the inverse limit space associated with $\mathbf{K}$ on V.

For $s \in V$, let $(A_s^i)_{i \in \mathbb{N}} \in J(\mathbf{K})$ with $s \in A_s^i$ $\forall i \in \mathbb{N}$; the natural embedding $\nu: V \to S(\mathbf{K})$ is then defined as in Definition 1.8 by $\nu(s) := (\varphi_i(A_s^i))_{i \in \mathbb{N}}$. Set $\mathscr{V} := \sigma\{\{x \in V | x \geqslant s\} | s \in V\}$;

Theorem 4.2. *$\nu: (V, \mathscr{V}) \to (S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ is a measurable injective map which is order-preserving. Furthermore,*

(i) $\nu(s) = (s_i)_{i \in \mathbb{N}} \Rightarrow s = \bigvee_{i \in \mathbb{N}}^{V} s_i$;

(ii) $(s_i)_{i \in \mathbb{N}} \in S(\mathbf{K}) \Rightarrow (s_i)_{i \in \mathbb{N}} = \bigvee_{j \in \mathbb{N}}^{S(\mathbf{K})} \nu(s_j)$.

Proof. $\nu^{-1}\{(s_i)_{i \in \mathbb{N}} \in S(\mathbf{K}) | s_n = s\} = \varphi_n^{-1}(s)$, so ν is measurable. If

$\nu(s) = (s_i)_{i\in\mathbb{N}}$, then $s \geqslant s_i$, and since there exists a sequence $(u_i)_{i\in\mathbb{N}}$ with $u_i \uparrow s$, $u_i \in K_{n(i)}$, we have $s \in A^{n(i)}_{s_{n(i)}} \subseteq \{x \in V | x \geqslant u_i\}$, which proves (i). So ν is injective, and $\nu(s) \geqslant \nu(t) \Rightarrow s \geqslant t$. Conversely, $s \geqslant t \geqslant t_i$ implies $s \in A^i_{s_i} \subseteq \{x \in V | x \geqslant t_i\}$, i.e., $\nu(s) = (s_i)_{i\in\mathbb{N}} \geqslant \nu(t) = (t_i)_{i\in\mathbb{N}}$. Q.E.D.

Examples

In Example 1, the most natural index set would be $V = [0, \infty)^n$ with $V^d = D^n$; in Example 2, $V = \{U | U \text{ an open set in } T\}$ with V^d a countable basis closed with respect to finite unions, whereas in Examples 3 and 4, V could be any $\vee$-semilattice or tree with a countable dense set V^d_1; the set V^d_1 can then always be completed to an exhaustible set V^d. ∎

Let $(\Omega, \mathscr{F}, P)$ be a probability space with an isotone family of sub-σ-algebras $(\mathscr{F}_s)_{s\in V}$ of $\mathscr{F}$ and $Q = (Q_s)_{s\in V}$ a set of measures on $(\Omega, \mathscr{F}_s)_{s\in V}$. If there is to exist a measure P^Q satisfying (4.1), then $(Q_s)_{s\in V^d}$ must be an E-quasimartingale on $\mathbf{K}$, and Q as well as $(\mathscr{F}_s)_{s\in V}$ must be consistent in the sense of Corollary 2.6: $Q_s = Q_{\nu(s)}$, $\mathscr{F}_s = \mathscr{F}_{\nu(s)}$, $\forall s \in V$.

ASSUMPTION 4.3. $(\mathscr{F}_s)_{s\in V}$ is an isotone family of sub-σ-algebras of $\mathscr{F}$ satisfying the Assumption 2.3 as well as

(iii) $\mathscr{F}_s = \sigma(\bigcup_{i\in\mathbb{N}} \mathscr{F}_{s_i})$, $\forall s \in V \backslash V^d$, where $(s_i)_{i\in\mathbb{N}} = \nu(s)$.

DEFINITION 4.4. $Q = (Q_s)_{s\in V}$ is an E-quasimartingale (of bounded variation) on $\mathbf{K}$ if

(i) $(Q_s)_{s\in V^d} =: Q^d$ is an E-quasimartingale (of bounded variation) on $\mathbf{K}$; $\mathbf{K}\text{-sv}(Q) := \mathbf{K}\text{-sv}(Q^d)$, $\mathbf{K}\text{-var}(Q) := \mathbf{K}\text{-var}(Q^d)$.

(ii) Q is weakly leftcontinuous, i.e., $\text{weak-lim}_{i\to\infty} Q_{s_i}(F) = Q_s(F)$, $\forall s \in V \backslash V^d$, $\forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}_{s_i}$, where $(s_i)_{i\in\mathbb{N}} = \nu(s)$.

A map $f: V \to E$ is of bounded (semi-) variation on $\mathbf{K}$ if Q defined on $\mathscr{F}_s = \{\varnothing, \Omega\}$ by $Q_s(\Omega) := f(s)$ is an E-quasimartingale (of bounded variation) on $\mathbf{K}$.

Note first that $\text{weak-lim}_{i\to\infty} Q_{s_i}(F) = Q_s(F)$ implies $\text{norm-lim}_{i\to\infty} Q_{s_i}(F) = Q_s(F)$ for an E-quasimartingale on $\mathbf{K}$ since $\sup\{\|Q_{s_i}(F)\| \,|\, i \in \mathbb{N}\} < \infty$, and, second, that we do not assume this leftcontinuity for all $s \in V$.

THEOREM 4.5. *Under Assumption* 4.3 *the statements of the Theorems* 2.4 *through Theorem* 2.9 *remain true for a separable poset* V.

This is an immediate consequence of Definition 4.4. For an adapted L^1-process $X = (X_s)_{s\in V}$ however, the situation is slightly different due to Lemma 2.12: if $X^d := (X_s)_{s\in V^d}$ is a quasimartingale on $\mathbf{K}$, $(X_s \cdot P)_{s\in V^d}$ may define measures $(Q_s)_{s\in S(K)}$ which are not absolutely continuous with respect to P anymore and therefore cannot be represented by $(X_s \cdot P)_{s\in S(K)}$.

DEFINITION 4.6. X is a quasimartingale on $\mathbf{K}$ if

(i) X^d is a quasimartingale on $\mathbf{K}$; $\mathbf{K}\text{-var}(X) := \mathbf{K}\text{-var}(X^d)$;

(ii) X is weakly a.s.-leftcontinuous, i.e., $\lim_{i\to\infty} X_{s_i} = X_s$ a.s. $\forall s \in V\backslash V^d$, where $(s_i)_{i\in\mathbb{N}} := v(s)$.

If in addition $\lim_{i\to\infty} E[X_{s_i}; F] = E[X_s; F]$, $\forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}_{s_i}$, $\forall s \in V\backslash V^d$, where $(s_i)_{i\in\mathbb{N}} = v(s)$, then $L^1 - \lim_{i\to\infty} X_{s_i} = X_s$, $\forall s \in V\backslash V^d$, and X is called weakly L^1-leftcontinuous.

THEOREM 4.7. *An adapted L^1-process X is a quasimartingale on* $\mathbf{K}$ *iff there exists a finite signed measure P^X on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ with $\|P^X\| =$ K-var(Q) and such that*

$$P^X[\{x \in S(\mathbf{K}) \mid x \geqslant v(s)\} \times F] = E[X_s; F] + \mu_s(F), \qquad \forall s \in V, \forall F \in \mathscr{F}_s,$$

where μ_s is a measure on $\mathscr{F}_s$ orthogonal to $P|_{\mathscr{F}_s}$, with $\mu_s \equiv 0\ \forall s \in V^d$. X is weakly L^1-leftcontinuous iff $\mu_s \equiv 0$, $\forall s \in V$.

Proof. This follows from Lemma 2.12. Q.E.D.

Examples

In Example 1, with $V = [0, \infty)$, the functions of bounded variation on $\mathbf{K} := (D_i^1)_{i\in\mathbb{N}}$ are precisely those functions of bounded variation (in the classical sense) which are leftcontinuous on $[0, \infty)\backslash D^1$. If $(\mathscr{F}_s)_{s\in[0,\infty)}$ is the leftcontinuous version of a set $(\mathscr{F}_s^0)_{s\in[0,\infty)}$ of σ-algebras satisfying Assumption 2.3, the leftcontinuous version of a positive $(\mathscr{F}_s^0)_{s\in[0,\infty)}$-adapted supermartingale is a quasimartingale on $\mathbf{K}$.

In Example 2, let $(X_z)_{z\in T}$ be a random field and μ a measure on $(T, \mathscr{B}(T))$. Define $\mathscr{F}_U := \sigma\{X_z \mid z \in U\}$ for the open sets $U \in V$ and $Y_U(\omega) := \int_U f(X_z(\omega))\, \mu(dz)$ for a measurable function f satisfying $\int_T E[|f(X_z)|]\, \mu(dz) < \infty$. If $(\mathscr{F}_U)_{U\in V}$ satisfies Assumption 2.3, then $Y = (Y_U)_{U\in V}$ is a quasimartingale on $\mathbf{K}$ with measure P^Y. Since for $U \in K_i$, $D_U^i Y = 0$ if there exists a $U' \in K_i$ with $U \subset U' \subset T$, i.e., if $|T\backslash U| \geqslant 2$, P^Y is concentrated on $\bigcap_{i\in\mathbb{N}} (\bigcup_{U\in K_i, |T\backslash U| = 1} \{U' \in S(\mathbf{K}) \mid U' \geqslant v(U)\}) \times \Omega$. ∎

To continue the process of generalization to separable V, let us also admit separable V in the definitions of Section 3.

THEOREM 4.8. *Under Assumption* 4.3, *all results of Section* 3 *remain correct if V is a separable poset.*

This follows easily since the proofs in Section 3 depend only on the measures P^Q associated with the quasimartingales Q.

One last remark on the inverse limit space $S(\mathbf{K})$: since $S(\mathbf{K})$ is itself a separable poset, as we will show now, the above constructions could be iterated; but this does not lead to any improvement:

LEMMA 4.9. *$S(\mathbf{K})$ is a separable poset with regularly separating sequence $\nu(\mathbf{K}) := (\nu(K_i))_{i\in\mathbb{N}}$. $(S(\nu(\mathbf{K})), \mathscr{S}(\nu(\mathbf{K})))$ and $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}))$ are σ-isomorphic measurable spaces.*

Proof. $\bigcup_{i\in\mathbb{N}} \nu(K_i)$ is dense in $S(\mathbf{K})$ since $S(\mathbf{K}) \ni (s_i)_{i\in\mathbb{N}} = \bigvee_{i\in\mathbb{N}} \nu(s_i)$ by Theorem 4.2, and $\nu(K_i)$ is $\bigcap$-compatible because K_i is. If $V = \bigcup_{s\in K_1} \{x \in V | x \geqslant s\}$, then

$$S(\nu(\mathbf{K})) = \{(\nu(s_i))_{i\in\mathbb{N}} | \nu(s_i) \leqslant \nu(s_{i+1}), \nu(s_i) \in \nu(K_i);$$

$$t \in \nu(K_i),\ t \nleqslant \nu(s_i) \Rightarrow t \nleqslant \nu(s_{i+1})\}$$

$$= \{(\nu(s_i))_{i\in\mathbb{N}} | s_i \leqslant s_{i+1}, s_i \in K_i;\ t \in K_i, t \nleqslant s_i \Rightarrow t \nleqslant s_{i+1}\}$$

so that $\psi: S(\nu(\mathbf{K})) \to S(\mathbf{K})$ defined by $\psi((\nu(s_i))_{i\in\mathbb{N}}) := (s_i)_{i\in\mathbb{N}}$ is bijective and is obviously also a σ-isomorphism. Similarly, if $V \neq \bigcup_{s\in K_1} \{x \in V | x \geqslant s\}$.

Q.E.D.

5. THE CASE $V = \mathbb{R}_+^n$

The assumptions on the σ-algebras $(\mathscr{F}_s)_{s\in V}$ and the quasimartingales Q which we have been making so far do not agree with the situation in the theory of stochastic processes for $V = [0, \infty)^n$, as it is found for example in [9, 10]. However, there is a simple canonical translation, as we will now show: in essence, change from rightcontinuous to leftcontinuous versions (in the appropriate topology).

Let $[0, \infty]^n$ be furnished with the natural order $\leqslant$ and the strengthening $\ll$ defined by $s = (s_1, ..., s_n) \ll t = (t_1, ..., t_n)$ iff $s_i < t_i$, $i = 1, ..., n$; $(s, t] := \{x \in [0, \infty]^n | s \ll x \leqslant t\}$; for a sequence $(s^i)_{i\in\mathbb{N}}$ in $[0, \infty]^n$ with $\lim_{i\to\infty} s^i = s$ we will write $s^i \uparrow s$ if $s^i \ll s^{i+1}$, and $s^i \downarrow s$ if $s^i \gg s^{i+1} \forall i \in \mathbb{N}$. As usual, $(\Omega, \mathscr{F}, P)$ is a probability space with an isotone set $(\mathscr{F}_s)_{s\in(0,\infty)^n}$ of sub-σ-algebras such that $(\Omega, \mathscr{F}_s)$ is a standard Borel space $\forall s \in (0, \infty)^n$; but now we assume that for any increasing sequence $(s^i)_{i\in\mathbb{N}}$ in $(0, \infty)^n$ there does not exist a decreasing sequence $(A_i)_{i\in\mathbb{N}}$ of atoms $A_i \in \mathscr{F}_{s^i}$ with $\bigcap_{i\in\mathbb{N}} A_i = \varnothing$ (this is the situation in [8–10]). Define the right- and leftcontinuous modifications of $(\mathscr{F}_s)_{s\in(0,\infty)^n}$ by

$$\mathscr{F}_s^r := \lim_{t \downarrow s} \mathscr{F}_t, \qquad \forall s \in [0, \infty)^n,$$

$$\mathscr{F}_s^l := \sigma\left(\lim_{t \uparrow s} \mathscr{F}_t\right), \quad \forall s \in (0, \infty]^n.$$

DEFINITION 5.1. An $(\mathscr{F}_s^r)_{s \in [0,\infty)^n}$ – adapted process $X = (X_s)_{s \in [0,\infty)^n}$ is a right-quasimartingale if

(i) $\sup\{\sum_{s \in K_i} E[|E[D_s^i X | \mathscr{F}_s^r]|] \mid i \in \mathbb{N}\} =: \mathscr{F}^r\text{-var}(X) < \infty$, where D_s^i is the upper difference operator on $K_i := \{(s_1, ..., s_n) 2^{-i} \mid s_j \in \{0, ..., i\, 2^i\} \subseteq \mathbb{N}, 1 \leqslant j \leqslant n\}$;

(ii) X is weakly L^1-rightcontinuous, i.e., $\lim_{t \downarrow s} X_t = X_s$ in the weak-$L^1(\Omega, \mathscr{F}_s)$ topology.

The right-quasimartingales are precisely the quasimartingales of Föllmer [10], and therefore they define a unique measure $P^{X,r}$ on the space $((0, \infty]^n \times \Omega, \mathscr{P})$, $\mathscr{P} := \sigma\{(s, \infty] \times F_s \mid F_s \in \mathscr{F}_s^r,\, s \in [0, \infty)^n\}$ the σ-algebra of the predictable sets, such that

$$P^{X,r}[(s, \infty] \times F_s] = E[X_s; F_s], \qquad \forall F_s \in \mathscr{F}_s^r.$$

DEFINITION 5.2. A set $Q = (Q_s)_{s \in (0,\infty]^n}$ of signed measures Q_s on $(\Omega, \mathscr{F}_s^l)$ is called a left-quasimartingale (of measures) if

(i) $\sup\{\sum_{s \in K_i} D_s^i Q\,|_{\mathscr{F}_s^l} \mid i \in \mathbb{N}\} =: \mathscr{F}^l\text{-var}(Q) < \infty$;

(ii) Q is weakly L^1-leftcontinuous, i.e., $\lim_{t \uparrow s} Q_t(F) = Q_s(F)$, $\forall F \in \bigcup_{t \ll s} \mathscr{F}_t^l$.

According to Theorems 2.7 and 4.5, a left-quasimartingale defines a measure $P^{Q,l}$ on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$. But since $S(\mathbf{K})$ is equivalent to the disjoint union union of $(0, \infty]^n$ with the countable set of hyperplanes $H_{c,I} := \{(s_1, ..., s_n) \in (0, \infty]^n \mid s_i = c_i, \forall i \subseteq I\}$, $c \in \bigcup_{i \in \mathbb{N}} K_i$, $I \subsetneq \{1, ..., n\}$, and since each set $H_{c,I} \times \Omega$ is a $P^{Q,l}$-nullset due to the weak L^1-leftcontinuity of Q, we may consider $P^{Q,l}$ as a measure on $((0, \infty]^n \times \Omega, \mathscr{P})$, where $\mathscr{P} = \sigma\{[s, \infty] \times F \mid F \in \mathscr{F}_s^l,\, s \in (0, \infty]^n\}$.

THEOREM 5.3. *The adapted process $X = (X_s)_{s \in [0,\infty)^n}$ is a right-quasimartingale iff there exists a left-quasimartingale $Q = (Q_s)_{s \in (0,\infty]^n}$ such that*

$$\lim_{t \uparrow s} E[X_t; F] = Q_s(F), \qquad \forall F \in \textstyle\bigcup_{t \ll s} \mathscr{F}_t^r, \forall s \in (0, \infty]^n, \tag{5.1}$$

$$\lim_{t \downarrow s} Q_t(F) = E[X_s; F], \qquad \forall F \in \mathscr{F}_s^r, \forall s \in [0, \infty)^n, \tag{5.2}$$

and then $P^{X,r} = P^{Q,l}$. *X defines Q uniquely by* (5.1) *and Q defines X uniquely by* (5.2).

Proof. If X is a right-quasimartingale, $\lim_{t\uparrow s} E[X_t; F] = P^{X,r}[[s, \infty] \times F] =: Q_s(F)$, $\forall F \in \bigcup_{t \ll s} \mathscr{F}^r_t$ defines a measure on $\sigma(\bigcup_{t \ll s} \mathscr{F}^r_t) = \mathscr{F}^l_s$, and the statements of the theorem follow. Inversely, if Q is a left-quasimartingale, the theorem follows from $X_s \cdot P(F) := P^{Q,l}[(s, \infty] \times F] = \lim_{t\downarrow s} Q_t(F)$, $\forall F \in \mathscr{F}^r_s$. Q.E.D.

For a right-quasimartingale X the classical theorem on convergence backwards implies that the weak L^1-rightcontinuity is equivalent with a kind of sequential a.s.-rightcontinuity, i.e., with $\lim_{i\to\infty} X_{s^i} = X_s$ a.s. $\forall (s^i)_{i\in\mathbb{N}}$ with $s^i \downarrow s$, $\forall s \in [0, \infty)^n$ (note that the set $\{\lim_{i\to\infty} X_{s^i} \neq X_s\}$ may depend on $(s^i)_{i\in\mathbb{N}}$). For left-quasimartingales the two conditions lead to different situations: weak L^1-continuity is appropriate for measures $Q = (Q_s)_{s\in(0,\infty]^n}$, whereas sequential a.s.-leftcontinuity is adequate for processes X:

DEFINITION 5.4. An $(\mathscr{F}^l_s)_{s\in(0,\infty]^n}$-adapted process $X = (X_s)_{s\in(0,\infty]^n}$ is a left-quasimartingale (of random variables) if

(i) X is sequentially a.s.-leftcontinuous, i.e., $\lim_{i\to\infty} X_{s^i} = X_s$ a.s. $\forall (s^i)_{i\in\mathbb{N}}$ with $s^i \uparrow s$, $\forall s \in (0, \infty]^n$;

(ii) There exists a regularly separating sequence $\mathbf{K} = (K_i)_{i\in\mathbb{N}}$ on $(0, \infty]^n$ such that $(\mathscr{F}^1, \mathbf{K})\text{-var}(X) := \sup\{\sum_{s\in K_i} \| E[D^i_s X | \mathscr{F}_s] \|_{L^1} | i \in \mathbb{N}\} < \infty$.

Denote $\inf\{(\mathscr{F}^1, \mathbf{K})\text{-var}(X) | \mathbf{K}$ a regularly separating sequence of $(0, \infty]^n\}$ by $\mathscr{F}^1\text{-var}(X)$.

To a left-quasimartingale X of random variables we could associate a measure P^X on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ by means of Theorem 4.7, where P^X is based on the $\mathbb{R}$-quasimartingale $(X_s \cdot P)_{s\in\bigcup_{i\in\mathbb{N}} K_i}$. However, P^X might depend on $\mathbf{K}$. This can be avoided if we make use of the strengthening $\ll$ to define a more natural measure $P^{X,l}$ on $((0, \infty]^n \times \Omega, \mathscr{P})$ in the following way:

THEOREM 5.5. *An* $(\mathscr{F}^l_s)_{s\in(0,\infty]^n}$*-adapted process* $X = (X_s)_{s\in(0,\infty]^n}$ *is a left-quasimartingale iff there exists a left-quasimartingale* $Q = (Q_s)_{s\in(0,\infty]^n}$ *of measures with*

$$Q_s(F) = \lim_{t\uparrow s} E[X_t; F], \qquad \forall F \in \bigcup_{t \ll s} \mathscr{F}^l_t, \tag{5.3}$$

$$Q_s = X_s \cdot P + \mu_s \qquad \forall s \in (0, \infty]^n, \tag{5.4}$$

where μ_s *is a measure on* $(\Omega, \mathscr{F}^l_s)$ *orthogonal to* $P|_{\mathscr{F}^l_s}$ *and* $\mu_s|_{\mathscr{F}^l_t}$ *is absolutely continuous with respect to* $P|_{\mathscr{F}^l_t}$, $\forall t \ll s$. *Then* $\mathscr{F}^l\text{-var}(Q) = \mathscr{F}^l\text{-var}(X)$.

The measure on $((0, \infty]^n \times \Omega, \mathscr{P})$ associated with this Q will be denoted by $P^{X,1}$.

Proof. Let X be a left-quasimartingale and let $\mathbf{K}$ be such that $(\mathscr{F}^l, \mathbf{K})$-$\operatorname{var}(X) < \infty$. If $(s^i)_{i \in \mathbb{N}} \subseteq \bigcup_{j \in \mathbb{N}} K_j$ with $s^i \uparrow s$, Lemma 2.12 applied to the quasimartingale $(X_{s^i})_{i \in \mathbb{N}}$ proves that (5.3) defines a measure Q_s on $\mathscr{F}^l_s$, which in fact does not depend on the particular sequence $(s^i)_{i \in \mathbb{N}} \subseteq \bigcup_{j \in \mathbb{N}} K_j$. Again by Lemma 2.12 $Q_s = X_s \cdot P + \mu_s$, where μ_s has the stated properties; so if $(u^i)_{i \in \mathbb{N}}$ is any sequence in $(0, \infty]^n$ with $u^i \uparrow u$,

$$
\begin{aligned}
(\mathscr{F}^l, \mathbf{K})\text{-}\operatorname{var}(X) &\geqslant \sup \Big\{ \sum_{i \in \mathbb{N}} E[|E[X_{t^{i+1}} - X_{t^i} | \mathscr{F}^l_{t^i}]|] \,\Big|\, t^i \in \bigcup_{j \in \mathbb{N}} K_j, \\
&\qquad\qquad u^i \ll t^{i+1} \ll u^{i+1}, \forall i \in \mathbb{N} \Big\} \\
&\geqslant \sum_{i \in \mathbb{N}} \| (Q_{u^{i+1}} - Q_{u^i})|_{\mathscr{F}^l_{u^i}} \| \qquad \text{due to (5.3)} \\
&= \sum_{i \in \mathbb{N}} \left\| \left(E[X_{u^{i+1}} - X_{u^i} | \mathscr{F}^l_{u^i}] + \frac{d\mu_{u^{i+1}}}{dP}\Big|_{\mathscr{F}^l_{u^i}} \right) \cdot P - \mu_{u^i} \right\| \\
&\geqslant \sum_{i \in \mathbb{N}} \|\mu_{u^i}\|,
\end{aligned}
$$

i.e., $\|\mu_{u^i}\| \to 0$ for $i \to \infty$, and this proves Q to be weakly L^1-leftcontinuous. From this (5.3) follows, and since $\mathscr{F}^l$-$\operatorname{var}(Q) \leqslant (\mathscr{F}^l, \mathbf{K})$-$\operatorname{var}(X)$, Q is a left-quasimartingale.

Now assume Q to be a left-quasimartingale of measures with $Q_s = X_s \cdot P + \mu_s$. The weak-$L^1$-leftcontinuity of Q together with $\sum_{i \in \mathbb{N}} \|\mu_{u^i}\| < \infty$ for $u^i \uparrow u$ implies (5.3), and together with Lemma 2.12 applied to the quasimartingale $(X_{u^i})_{i \in \mathbb{N}}$ this proves X to be sequentially a.s.-leftcontinuous. So it only remains to show that $\mathscr{F}^l$-$\operatorname{var}(X) = \mathscr{F}^l$-$\operatorname{var}(Q)$, and this follows immediately if we can find a regularly separating sequence $\mathbf{K}^1$ with $Q_s = X_s \cdot P$, $\forall s \in \bigcup_{i \in \mathbb{N}} K^1_i$.

For $s \in (0, \infty]^n, F \in \mathscr{F}^l_s$, $\lambda(A) := P^Q[A \times F]$ defines a finite signed measure on $([s, \infty], \mathscr{B}[s, \infty])$. The set

$$
I_f := \bigcup_{i=1}^{n} \{ c \in (0, \infty]^n \mid |\lambda| \{ (s_1, \ldots, s_n) | s_i = c \} > 0 \}
$$

is countable, and for $t \in [s, \infty] \setminus \prod_{i=1}^n I_F$ we have

$$
\begin{aligned}
E[X_t;F]+\mu_t(F) &= \lambda([t,\infty]) \\
&= \lim_{u\downarrow t}\lambda([u,\infty]) \\
&= \lim_{u\downarrow t} Q_u(F) \\
&= E[X_t^r;F]
\end{aligned}
$$

by Theorem 5.3, where X^r is the right-quasimartingale with $P^{X^r,r}=P^{Q,l}$. Since $\mathscr{F}_s$ is countably generated and $(\mathscr{F}_s)_{s\in(0,\infty)^n}$ is isotone, there exists a countable ring $\mathscr{R}$ such that $\forall s\in(0,\infty)^n$, $\mathscr{R}$ contains a ring $\mathscr{R}_s$ with $\sigma(\mathscr{R}_s)=\mathscr{F}_s$. $I:=\bigcup_{F\in\mathscr{R}} I_F$ is countable, and for $t\in(0,\infty]^n\setminus\prod_{i=1}^n I$ we have $E[X_t;F]+\mu_t(F)=E[X_t^r;F]$, $\forall F\in\bigcup_{s\ll t}\mathscr{R}_s$, and as μ_t is orthogonal to P on $\mathscr{F}_t^l=\sigma(\bigcup_{s\ll t}\mathscr{R}_s)$, μ_t must vanish: choosing $\mathbf{K}^1\subseteq(0,\infty]^n\setminus\prod_{i=1}^n I$, we are finished. Q.E.D.

The proof shows that $P^{X,1}$ is the measure associated with the R-quasimartingale on $\mathbf{K}^1$ by Theorem 4.7, where $\mathbf{K}^1$ is carefully chosen; this dependence on the sequence $\mathbf{K}$ does not occur for right-quasimartingales. But in spite of this difficulty, there exists a complete duality between right- and left-quasimartingales: call $K:=\{(s_1,\dots,s_n)\mid s_i\in\{c_1,\dots,c_m\},\ 1\leqslant i\leqslant n\}$ a grid with upper difference operator D^K; then we have

THEOREM 5.6. *Let $X=(X_s)_{s\in(0,\infty)^n}$ be an $(\mathscr{F}_s)_{s\in(0,\infty)^n}$-adapted process with $X_s\equiv 0$ $\forall s\in(0,\infty]^n\setminus(0,\infty)^n$, and with $\mathscr{F}\text{-var}(X):=\sup\{\sum_{s\in K}\|E[D_s^K X|\mathscr{F}_s]\|\,|K \text{ a grid}\}<\infty$. For every $s\in[0,\infty]^n$ choose arbitrary sequences $(s^i)_{i\in N}$ and $(s_i)_{i\in N}$ with $s^i\uparrow s$ and $s_i\downarrow s$, and define*

$$X_s^r:=\lim_{i\to\infty} X_{s_i}\mathbf{1}_{\{\lim_{i\to\infty}X_{s_i}\text{exists}\}},\qquad \forall s\in[0,\infty)^n;$$

$$X_s^l:=\lim_{i\to\infty} X_{s^i}\mathbf{1}_{\{\lim_{i\to\infty}X_{s^i}\text{exists}\}},\qquad \forall s\in(0,\infty]^n;$$

$$Q_s(F):=\lim_{i\to\infty} E[X_{s^i};F],\qquad \forall F\in\bigcup_{t\ll s}\mathscr{F}_t,\ \forall s\in(0,\infty]^n.$$

Then $X^r=(X_s^r)_{s\in[0,\infty)^n}$ is a right-quasimartingale on $(\mathscr{F}_s^r)_{s\in[0,\infty)^n}$, $X^l=(X_s^l)_{s\in(0,\infty]^n}$ is a left-quasimartingale of random variables on $(\mathscr{F}_s^l)_{s\in(0,\infty]^n}$, and $Q=(Q_s)_{s\in(0,\infty]^n}$ is a left-quasimartingale of measures on $(\mathscr{F}_s^l)_{s\in(0,\infty]^n}$, with

$$P^{X^r,r}=P^{X^l,l}=P^{Q,l},$$

$$\mathscr{F}^r\text{-var}(X^r)=\mathscr{F}^l\text{-var}(X^l)=\mathscr{F}^l\text{-var}(Q)\leqslant\mathscr{F}\text{-var}(X).$$

In particular, X^r and X^l are sequentially a.s.-rightcontinuous and leftcontinuous versions of X. If $E[D_s^K X|\mathscr{F}_s] \geqslant 0$, $\forall s \in K$, $\forall$ grids K, X is sequentially a.s.-rightcontinuous (leftcontinuous) iff X is $\mathscr{F}^r$- ($\mathscr{F}^l$-) adapted and $f(s) := E[X_s]$ is a rightcontinuous (leftcontinuous) function because $(X_{s^i})_{i \in \mathbb{N}}$ is a positive supermartingale for any increasing sequence $(s^i)_{i \in \mathbb{N}}$.

Proof. If $(s^i)_{i \in \mathbb{N}}$ is a monotone sequence, $(X_{s^i})_{i \in \mathbb{N}}$ is a quasimartingale, so X^r, X^l and Q are defined. Since $X_{s_i} \to X_s^r$ in $L^1(P)$, X^r is a right-quasimartingale with $\mathscr{F}^r\text{-var}(X^r) \leqslant \mathscr{F}\text{-var}(X)$. Let Q^l be the left-quasimartingale associated with X^r by Theorem 5.3; if $s^i \uparrow s$, let $(u_j^i)_{i,j \in \mathbb{N}}$ be such that $s^i \ll u_j^i \ll s^{i+1}$ and $u_j^i \downarrow s^i$ for $j \to \infty$. Then $(X_{u_j^i})_{j \in \mathbb{N}}$ converges to $X_{s^i}^r$ a.s. and in $L^1(P)$, so that $\sup_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \|E[X_{u_j^i} - X_{s^i}|\mathscr{F}_{s^i}]\| < \infty$ implies $\sum_{i \in \mathbb{N}} \|X_{s^i}^r - X_{s^i}\| < \infty$, and therefore $Q_s = Q_s^l$, which proves that Q is a left-quasimartingale. Since $Q_s = X_s^l \cdot P + \mu_s$, μ_s orthogonal to P on $\mathscr{F}_s^l$, X^l is a left-quasimartingale by the last theorem. Q.E.D.

Example

Let $X = (X_s)_{s \in [0,\infty)^2}$ be an $(\mathscr{F}_s^r)_{s \in [0,\infty)^2}$-adapted martingale with $\mathscr{F}^r$-var$(X) < \infty$ and let Q be the associated left-quasimartingale of measures on $(\Omega, \mathscr{F}_s^l)_{s \in (0,\infty]^2}$. Assume furthermore that X vanishes on the axes and set $\mathscr{F}_s^l := \mathscr{F}_s^r$ and $Q_s := 0$ on the axes; then $Q = (Q_s)_{s \in [0,\infty]^2}$ is a martingale in the sense of Definition 3.4. If $V^d := \{[0,\infty)^2 \setminus \bigcup_{i=1}^m [0, s^i) | s^i \in \mathbb{Q}_+^2$, $1 \leqslant i \leqslant m$, $m \in \mathbb{N}\}$, then V^d ordered by $C_1 \geqslant C_2 \Leftrightarrow C_1 \subseteq C_2$ is exhaustible, and every $C \in V^d$ can be written as $C = \bigcup_{i=1,\dots,k}^d [s^{1i}, s^{2i})$; for such a C we define

$$Q_C := \sum_{i=1}^{k} Q[s^{1i}, s^{2i}),$$

where $Q[s,t) := Q_s - Q_{(s_1,t_2)} - Q_{(t_1,s_2)} + Q_t$, and

$$\mathscr{F}_C^l := \bigvee_{i=1}^{m} \mathscr{F}_{s^i}^l$$

for $C = [0,\infty)^2 \setminus \bigcup_{i=1}^m [0, s^i)$. Then $(Q_C)_{C \in V^d}$ is a martingale with respect to $(\mathscr{F}_C^l)_{C \in V^d}$ in the sense of Definition 3.4 iff X is a strong martingale, i.e., iff $E[X(s,t]|\sigma(\bigcup_{u \not\geqslant s} \mathscr{F}_u^r)] = 0$, $\forall s \ll t \in [0,\infty)^2$, where $X(s,t]$ is defined in the same way as $Q[s,t)$. If

$$V := \{C \subseteq [0,\infty)^2 | s \in C \Rightarrow [s,\infty) \subseteq C;\ C \cap [0,t] \text{ is compact } \forall t \ll \infty\},$$

then V^d is dense in V, and $(Q_C)_{C \in V^d}$ can be extended to a martingale on V by defining

$$\mathscr{F}^l_C := \bigcap\{\mathscr{F}^l_D \mid D \in V^d, \quad C\ \mathrm{int}(D)\}$$

$$Q_C(F) := \lim_{i\to\infty} Q_{D_i}(F) \text{ for } (D_i)_{i\in\mathbb{N}} \subseteq V^d \text{ with } \mathrm{int}(D_i) \downarrow C, \qquad \forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}^l_{D_i}$$

$$= P^Q[\{D \in S(\mathbf{K}) \mid D \geqslant \nu(C)\} \times F],$$

where P^Q is the measure on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ associated with Q, $\mathbf{K}$ a regularly exhausting sequence of V^d; P^Q exists and is concentrated on $\{1\} \times \Omega$, 1 the largest element in $S(\mathbf{K})$.

6. Uniqueness of the Associated Measures

If V is a fixed separable poset, we now want to show in which sense the inverse limit space and the associated measures constructed in the preceding paragraphs are independent of the chosen regularly separating sequence. So let $\mathbf{K} = (K_i)_{i\in\mathbb{N}}$ add $\mathbf{L} = (L_i)_{i\in\mathbb{N}}$ be arbitrary regularly separating sequences; if necessary, we shall and an index $\mathbf{K}$ or $\mathbf{L}$ to the standard notations to distinguish the sequence we are referring to. As usual, let Q be a set of σ-additive measures Q_s on $(\Omega, \mathscr{F}_s)$ with values in a Banach space E. The main results are then given in Theorems 6.1 and 6.11 below.

Theorem 6.1. *Let V be exhaustible with regularly exhausting sequences $\mathbf{K}$ and $\mathbf{L}$, and let $(\mathscr{F}_s)_{s\in V}$ satisfy Assumption 2.3 with respect to $\mathbf{K}$. Then $(\mathscr{F}_s)_{s\in V}$ also satisfy Assumption 2.3 with respect to $\mathbf{L}$, and*

(i) *there exists a map $\tilde{\Psi}$: $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}), \geqslant) \to (S(\mathbf{L}), \mathscr{S}(\mathbf{L}), \geqslant)$ which is an isomorphism of measurable spaces and of posets;*

(ii) $\mathbf{K}$-sv$(Q) = \mathbf{L}$-sv(Q); $\mathbf{K}$-var$(Q) = \mathbf{L}$-var(Q);

(iii) *$\tilde{\Psi}$ can be chosen in such a way that the map Ψ: $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K})) \to (S(\mathbf{L}) \times \Omega, \mathscr{P}(\mathbf{L}))$ defined by $\Psi\ (s, \omega) := (\tilde{\Psi}(s), \omega)$ is an isomorphism of measurable spaces with the property that ${}^{\mathbf{K}}P^Q \circ \Psi^{-1} = {}^{\mathbf{L}}P^Q$ for every E-quasimartingale Q.*

Proof. If $\mathscr{K}_i := \sigma\{\{x \geqslant s\} \mid s \in K_i\}$ and $\mathscr{L}_i := \sigma\{\{x \geqslant s\} \mid s \in L_i\}$, there exist sequences $(m(i))_{i\in\mathbb{N}}$ and $(n(i))_{i\in\mathbb{N}}$ such that $\mathscr{K}_{m(i)} \subseteq \mathscr{L}_{n(i)} \subseteq \mathscr{K}_{m(i+1)}$, $\forall i \in \mathbb{N}$, since $V = \bigcup_{i\in\mathbb{N}} K_i = \bigcup_{i\in\mathbb{N}} L_i$. If $(A_i)_{i\in\mathbb{N}}$ is a decreasing sequence of atoms A_i of $\mathscr{K}_i$, there then exists a decreasing sequence of atoms B_i of $\mathscr{L}_i$ such that $A_{m(i)} \supseteq B_{n(i)} \supseteq A_{m(i+1)}$, $\forall i \in \mathbb{N}$, and $(B_i)_{i\in\mathbb{N}}$ is unique. This shows that $(\mathscr{F}_s)_{s\in V}$ satisfy Assumption 2.3 with respect to $\mathbf{K}$ iff they satisfy Assumption 2.3 with respect to $\mathbf{L}$. Denote the elements of $S(\mathbf{K})$ and $S(\mathbf{L})$ associated with $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ by $(s_i)_{i\in\mathbb{N}}$ and $(u_i)_{i\in\mathbb{N}}$, respectively, and sense that $s^i \uparrow s$ in V implies $Q_s(F) = \text{weak-lim}_{i\to\infty} Q_{s^i}(F)$, $\forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}_{s^i}$, then Q is regular.

(2) If Q is an E-quasimartingale on $\mathbf{K}$ satisfying (6.1), then ${}^{\mathbf{K}}Q\{x \geqslant s^1,..., s^k\}$ is defined, and the assumption

$$ {}^{\mathbf{K}}Q\{x \geqslant s^1,..., s^k\} = Q^{L_i}\{x \geqslant s^1,..., s^k\} \qquad \forall \{s^1,..., s^k\} \subseteq L_i \tag{6.2'} $$

suffices to prove (Lemma 6.8) that ${}^{\mathbf{L}}Q\{x \geqslant s^1,..., s^k\}$ is also defined $\forall \{s^1,..., s^k\} \subseteq V$. So the importance of (6.2) lies in the independence of $Q\{x \geqslant s^1,..., s^k\}$ from $\mathbf{K}$, and not in its existence.

(3) If Q is an S-process on $\mathbf{K}$ satisfying (6.1), then (6.2) and (6.2′) are equivalent. ∎

Let

${}^{\mathbf{K}}\rho^E := \{{}^{\mathbf{K}}P^Q \mid {}^{\mathbf{K}}P^Q$ is the measure on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ associated with Q, Q a regular E-quasimartingale on $\mathbf{K}\}$;

${}^{\mathbf{K}}\beta^E := \{{}^{\mathbf{K}}P^Q \in {}^{\mathbf{K}}\rho^E \mid Q$ is of bounded variation on $\mathbf{K}\}$;

${}^{\mathbf{K}}\rho^+ := \{{}^{\mathbf{K}}P^Q \in {}^{\mathbf{K}}\rho^R \mid Q$ is an S-process on $\mathbf{K}\}$;

${}^{\mathbf{K}}\chi := \{{}^{\mathbf{K}}P^X \in {}^{\mathbf{K}}\rho^R \mid X$ is an L^1-regular process with $\mathbf{K}\text{-var}(X) < \infty\}$;

THEOREM 6.6. *$(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}), {}^{\mathbf{K}}\rho^E)$ and $(S(\mathbf{L}) \times \Omega, \mathscr{P}(\mathbf{L}), {}^{\mathbf{L}}\rho^E)$ are σ-isomorph; the σ-isomorphism Ψ can be chosen in such a way that $\Psi'({}^{\mathbf{L}}P^Q) = {}^{\mathbf{K}}P^Q$ $\forall {}^{\mathbf{L}}P^Q \in {}^{\mathbf{L}}\rho^E$. This remains true if ${}^{\cdot}\rho^E$ is replaced by ${}^{\cdot}\rho^E$, ${}^{\cdot}\rho^+$ or ${}^{\cdot}\chi$.*

COROLLARY 6.7. *Let Q be regular; then*

$$ \mathbf{K}\text{-sv}(Q) = \mathbf{L}\text{-sv}(Q), \tag{6.3} $$

$$ \mathbf{K}\text{-var}(Q) = \mathbf{L}\text{-var}(Q). \tag{6.4} $$

If $\mathbf{K}\text{-sv}(Q) < \infty$ and ${}^{\cdot}P^Q$ are the associated measures, then $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}), {}^{\mathbf{K}}P^Q)$ and $(S(\mathbf{L}) \times \Omega, \mathscr{P}(\mathbf{L}), {}^{\mathbf{L}}P^Q)$ are isomorphic measure spaces in the sense of Halmos [11].

We assume that V has a smallest element 0, since otherwise the statements follow from $V \cup \{0\}$. The proof will be split into a number of lemmas.

If G is a $\bigcap$-compatible set in V, every atom A_s^G of $\mathscr{G} := \sigma\{\{x \geqslant t \mid t \in G\}$ can be written as

$$ A_s^G = \{x \in V \mid x \geqslant s\} \setminus \bigcup_{t >_G s} \{x \in V \mid x \geqslant t\}, $$

$$\mathscr{F}^l_C := \bigcap\{\mathscr{F}^l_D \mid D \in V^d, \quad C \operatorname{int}(D)\}$$

$$Q_C(F) := \lim_{i\to\infty} Q_{D_i}(F) \text{ for } (D_i)_{i\in\mathbb{N}} \subseteq V^d \text{ with } \operatorname{int}(D_i) \downarrow C, \qquad \forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}^l_{D_i}$$

$$= P^Q[\{D \in S(\mathbf{K}) \mid D \geqslant v(C)\} \times F],$$

where P^Q is the measure on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ associated with Q, $\mathbf{K}$ a regularly exhausting sequence of V^d; P^Q exists and is concentrated on $\{1\} \times \Omega$, 1 the largest element in $S(\mathbf{K})$.

6. Uniqueness of the Associated Measures

If V is a fixed separable poset, we now want to show in which sense the inverse limit space and the associated measures constructed in the preceding paragraphs are independent of the chosen regularly separating sequence. So let $\mathbf{K} = (K_i)_{i\in\mathbb{N}}$ add $\mathbf{L} = (L_i)_{i\in\mathbb{N}}$ be arbitrary regularly separating sequences; if necessary, we shall and an index $\mathbf{K}$ or $\mathbf{L}$ to the standard notations to distinguish the sequence we are referring to. As usual, let Q be a set of σ-additive measures Q_s on $(\Omega, \mathscr{F}_s)$ with values in a Banach space E. The main results are then given in Theorems 6.1 and 6.11 below.

Theorem 6.1. *Let V be exhaustible with regularly exhausting sequences $\mathbf{K}$ and $\mathbf{L}$, and let $(\mathscr{F}_s)_{s\in V}$ satisfy Assumption 2.3 with respect to $\mathbf{K}$. Then $(\mathscr{F}_s)_{s\in V}$ also satisfy Assumption 2.3 with respect to $\mathbf{L}$, and*

(i) *there exists a map $\tilde{\Psi}$: $(S(\mathbf{K}), \mathscr{S}(\mathbf{K}), \geqslant) \to (S(\mathbf{L}), \mathscr{S}(\mathbf{L}), \geqslant)$ which is an isomorphism of measurable spaces and of posets;*

(ii) $\mathbf{K}$-sv$(Q) = \mathbf{L}$-sv(Q); $\mathbf{K}$-var$(Q) = \mathbf{L}$-var(Q);

(iii) *$\tilde{\Psi}$ can be chosen in such a way that the map Ψ: $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K})) \to (S(\mathbf{L}) \times \Omega, \mathscr{P}(\mathbf{L}))$ defined by $\Psi\,(s, \omega) := (\tilde{\Psi}(s), \omega)$ is an isomorphism of measurable spaces with the property that ${}^{\mathbf{K}}P^Q \circ \Psi^{-1} = {}^{\mathbf{L}}P^Q$ for every E-quasimartingale Q.*

Proof. If $\mathscr{K}_i := \sigma\{\{x \geqslant s\} \mid s \in K_i\}$ and $\mathscr{L}_i := \sigma\{\{x \geqslant s\} \mid s \in L_i\}$, there exist sequences $(m(i))_{i\in\mathbb{N}}$ and $(n(i))_{i\in\mathbb{N}}$ such that $\mathscr{K}_{m(i)} \subseteq \mathscr{L}_{n(i)} \subseteq \mathscr{K}_{m(i+1)}$, $\forall i \in \mathbb{N}$, since $V = \bigcup_{i\in\mathbb{N}} K_i = \bigcup_{i\in\mathbb{N}} L_i$. If $(A_i)_{i\in\mathbb{N}}$ is a decreasing sequence of atoms A_i of $\mathscr{K}_i$, there then exists a decreasing sequence of atoms B_i of $\mathscr{L}_i$ such that $A_{m(i)} \supseteq B_{n(i)} \supseteq A_{m(i+1)}$, $\forall i \in \mathbb{N}$, and $(B_i)_{i\in\mathbb{N}}$ is unique. This shows that $(\mathscr{F}_s)_{s\in V}$ satisfy Assumption 2.3 with respect to $\mathbf{K}$ iff they satisfy Assumption 2.3 with respect to $\mathbf{L}$. Denote the elements of $S(\mathbf{K})$ and $S(\mathbf{L})$ associated with $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ by $(s_i)_{i\in\mathbb{N}}$ and $(u_i)_{i\in\mathbb{N}}$, respectively, and sense that $s^i \uparrow s$ in V implies $Q_s(F) = \text{weak-lim}_{i\to\infty} Q_{s^i}(F)$, $\forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}_{s^i}$, then Q is regular.

(2) If Q is an E-quasimartingale on $\mathbf{K}$ satisfying (6.1), then ${}^{\mathbf{K}}Q\{x \geqslant s^1,..., s^k\}$ is defined, and the assumption

$$ {}^{\mathbf{K}}Q\{x \geqslant s^1,..., s^k\} = Q^{L_i}\{x \geqslant s^1,..., s^k\} \qquad \forall \{s^1,..., s^k\} \subseteq L_i \tag{6.2$'$}$$

suffices to prove (Lemma 6.8) that ${}^{\mathbf{L}}Q\{x \geqslant s^1,..., s^k\}$ is also defined $\forall \{s^1,..., s^k\} \subseteq V$. So the importance of (6.2) lies in the independence of $Q\{x \geqslant s^1,..., s^k\}$ from $\mathbf{K}$, and not in its existence.

(3) If Q is an S-process on $\mathbf{K}$ satisfying (6.1), then (6.2) and (6.2′) are equivalent. ∎

Let

${}^{\mathbf{K}}\rho^E := \{{}^{\mathbf{K}}P^Q \mid {}^{\mathbf{K}}P^Q$ is the measure on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ associated with Q, Q a regular E-quasimartingale on $\mathbf{K}\}$;

${}^{\mathbf{K}}\beta^E := \{{}^{\mathbf{K}}P^Q \in {}^{\mathbf{K}}\rho^E \mid Q$ is of bounded variation on $\mathbf{K}\}$;

${}^{\mathbf{K}}\rho^+ := \{{}^{\mathbf{K}}P^Q \in {}^{\mathbf{K}}\rho^R \mid Q$ is an S-process on $\mathbf{K}\}$;

${}^{\mathbf{K}}\chi := \{{}^{\mathbf{K}}P^X \in {}^{\mathbf{K}}\rho^R \mid X$ is an L^1-regular process with $\mathbf{K}\text{-var}(X) < \infty\}$;

THEOREM 6.6. *$(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}), {}^{\mathbf{K}}\rho^E)$ and $(S(\mathbf{L}) \times \Omega, \mathscr{P}(\mathbf{L}), {}^{\mathbf{L}}\rho^E)$ are σ-isomorph; the σ-isomorphism Ψ can be chosen in such a way that $\Psi'({}^{\mathbf{L}}P^Q) = {}^{\mathbf{K}}P^Q$ $\forall {}^{\mathbf{L}}P^Q \in {}^{\mathbf{L}}\rho^E$. This remains true if ${}^{\cdot}\rho^E$ is replaced by ${}^{\cdot}\beta^E$, ${}^{\cdot}\rho^+$ or ${}^{\cdot}\chi$.*

COROLLARY 6.7. *Let Q be regular; then*

$$\mathbf{K}\text{-sv}(Q) = \mathbf{L}\text{-sv}(Q), \tag{6.3}$$

$$\mathbf{K}\text{-var}(Q) = \mathbf{L}\text{-var}(Q). \tag{6.4}$$

If $\mathbf{K}\text{-sv}(Q) < \infty$ and ${}^{\cdot}P^Q$ are the associated measures, then $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}), {}^{\mathbf{K}}P^Q)$ and $(S(\mathbf{L}) \times \Omega, \mathscr{P}(\mathbf{L}), {}^{\mathbf{L}}P^Q)$ are isomorphic measure spaces in the sense of Halmos [11].

We assume that V has a smallest element 0, since otherwise the statements follow from $V \cup \{0\}$. The proof will be split into a number of lemmas.

If G is a $\bigcap$-compatible set in V, every atom A_s^G of $\mathscr{G} := \sigma\{\{x \geqslant t \mid t \in G\}$ can be written as

$$A_s^G = \{x \in V \mid x \geqslant s\} \setminus \bigcup_{t \triangleright_G s} \{x \in V \mid x \geqslant t\},$$

where $t \gtrdot_G s$ means that t is a nearest neighbour of s in G, i.e., $G \ni t > s$, and $u \in G$ with $t \geqslant u > s$ implies $u = t$. Define

$$ {}^{\mathbf{K}}A_s^G := \{x \in S(\mathbf{K}) | x \geqslant \nu^{\mathbf{K}}(s)\} \setminus \bigcup_{t \gtrdot_G s} \{x \in S(\mathbf{K}) | x \geqslant \nu^{\mathbf{K}}(t)\}, $$

$$ {}^{\mathbf{K}}A^G := \bigcup_{i \in I} {}^{\mathbf{K}}A_{s_i}^G \qquad \text{for } A^G = \overset{d}{\bigcup_{i \in I}} A_{s_i}^G. $$

Based on the following technical result, the idea is to show that the map $\tilde{\Psi}: {}^{\mathbf{K}}A_s^{K_i} \to {}^{\mathbf{L}}A_s^{K_i}$ induces the σ-isomorphism of Theorem 6.6.

LEMMA 6.8. *Let Q be a regular E-quasimartingale on* $\mathbf{L}$; *if* $F_s \in \mathscr{F}_s$, $F_t \in \mathscr{F}_t$,

$$ {}^{\mathbf{L}}p^Q[{}^{\mathbf{L}}A_s^{K_i} \times F_s \cap {}^{\mathbf{L}}A_t^{K_i} \times F_t] = \begin{cases} 0 & \text{if } s \neq t \\ D_s^{K_i} Q(F_s) & \text{if } s = t, \end{cases} \tag{6.5} $$

$$ \mathrm{sv}({}^{\mathbf{L}}p^Q, {}^{\mathbf{L}}A_s^{K_i} \times F_s \cap {}^{\mathbf{L}}A_t^{K_j} \times F_t) = \mathrm{sv}({}^{\mathbf{K}}p^Q, {}^{\mathbf{K}}A_s^{K_i} \times F_s \cap {}^{\mathbf{K}}A_t^{K_j} \times F_t). \tag{6.6} $$

In particular, $\mathbf{K}$-$\mathrm{sv}(Q) = \mathbf{L}$-$\mathrm{sv}(Q)$ *and* $\mathbf{K}$-$\mathrm{var}(Q) = \mathbf{L}$-$\mathrm{var}(Q)$.

Proof. For $\{s^1, \dots, s^k\} \subseteq K_i$ and $F \in \mathscr{F}\{x \geqslant s^1, \dots, s^k\}$

$$ \begin{aligned} {}^{\mathbf{L}}p^Q[\{x \geqslant \nu^{\mathbf{L}}(s^1), \dots, \nu^{\mathbf{L}}(s^k)\} \times F] &= {}^{\mathbf{L}}Q\{x \geqslant s^1, \dots, s^k\}(F) \\ &= {}^{\mathbf{K}}Q\{x \geqslant s^1, \dots, s^k\}(F) \end{aligned} \tag{6.7} $$

due to (6.2′). Using the sieve formula (Theorem A.4) twice, we then get with $I := \{u \in K_i | (u \gtrdot_{K_i} s, u \geqslant t) \text{ or } (u \gtrdot_{K_i} t, u \geqslant s)\}$

$$ \begin{aligned} {}^{\mathbf{L}}p^Q[{}^{\mathbf{L}}A_s^{K_i} \cap {}^{\mathbf{L}}A_t^{K_i} \times F_s \cap F_t] &= {}^{\mathbf{L}}p^Q[\{x \geqslant \nu^{\mathbf{L}}(s), \nu^{\mathbf{L}}(t)\} \times F_s \cap F_t] \\ &\quad - {}^{\mathbf{L}}p^Q\left[\bigcup_{u \in I} \{x \geqslant \nu^{\mathbf{L}}(u)\} \times F_s \cap F_t\right] \\ &= Q^{K_i}\{x \geqslant s, t\}(F_s \cap F_t) \\ &\quad - Q^{K_i}\left(\bigcup_{u \in I} \{x \geqslant u\}\right)(F_s \cap F_t) \\ &= Q^{K_i}(A_s^{K_i} \cap A_t^{K_i})(F_s \cap F_t) \end{aligned} $$

which proves (6.5). As a consequence, if Π_s is a partition of $(\Omega, \mathscr{F}_s)$,

$$\sum_{s \in K_i} \sum_{F \in \Pi_s} \mathscr{E}_{s,F} D_s^{K_i} Q(F) = \sum_{s \in K_i} \sum_{F \in \Pi_s} \mathscr{E}_{s,F} {}^{\mathbf{L}}P^Q[{}^{\mathbf{L}}A_s^{K_i} \times F]$$
$$\leqslant \mathrm{sv}({}^{\mathbf{L}}P^Q, S(\mathbf{L}) \times \Omega) \qquad \text{if } |\mathscr{E}_{s,F}| \leqslant 1,$$

so that Q is an E-quasimartingale on $\mathbf{K}$, and, by symmetry, $\mathbf{K}\text{-sv}(Q) = \mathbf{L}\text{-sv}(Q)$; similarly, $\mathbf{K}\text{-var}(Q) = \mathbf{L}\text{-var}(Q)$.

So far, we have only used (6.2′); from (6.2) it follows that (6.7) is correct for any $\{s^1, \ldots, s^k\} \subseteq V$, and therefore

$${}^{\mathbf{L}}P^Q[\{x \geqslant \nu^{\mathbf{L}}(s^1), \ldots, \nu^{\mathbf{L}}(s^k)\} \times F] = {}^{\mathbf{K}}P^Q[\{x \geqslant \nu^{\mathbf{K}}(s^1), \ldots, \nu^{\mathbf{K}}(s^k)\} \times F] \tag{6.8}$$

for $F \in \mathscr{F}\{x \geqslant s^1, \ldots, s^k\}$. Applying the sieve formula as above,

$$\begin{aligned} &{}^{\mathbf{L}}P^Q[({}^{\mathbf{L}}A_s^{K_i} \cap {}^{\mathbf{L}}A_t^{K_j} \times F_s \cap F_t) \cap \{x \geqslant \nu^{\mathbf{L}}(s^1), \ldots, \nu^{\mathbf{L}}(s^k)\} \times F] \\ &= {}^{\mathbf{K}}P^Q[{}^{\mathbf{K}}A_s^{K_i} \cap {}^{\mathbf{K}}A_t^{K_j} \times F_s \cap F_t) \cap \{x \geqslant \nu^{\mathbf{K}}(s^1), \ldots, \nu^{\mathbf{K}}(s^k)\} \times F] \end{aligned} \tag{6.9}$$

follows, and this shows, again by the sieve formula, that

$$\begin{aligned} &{}^{\mathbf{L}}P^Q[({}^{\mathbf{L}}A_s^{K_i} \cap {}^{\mathbf{L}}A_t^{K_j} \times F_s \cap F_t) \cap {}^{\mathbf{L}}A_u^{L_m} \times F_u] \\ &= {}^{\mathbf{K}}P^Q[({}^{\mathbf{K}}A_s^{K_i} \cap {}^{\mathbf{K}}A_t^{K_j} \times F_s \cap F_t) \cap {}^{\mathbf{K}}A_u^{L_m} \times F_u] \end{aligned} \tag{6.10}$$

for $F_u \in \mathscr{F}_u$, which proves $\mathrm{sv}({}^{\mathbf{L}}P^Q, {}^{\mathbf{L}}A_s^{K_i} \cap {}^{\mathbf{L}}A_t^{K_j} \times F_s \cap F_t) \leqslant \mathrm{sv}({}^{\mathbf{K}}P^Q, {}^{\mathbf{K}}A_s^{K_i} \cap {}^{\mathbf{K}}A_t^{K_j} \times F_s \cap F_t)$. Equation (6.6) now follows by symmetry.

Q.E.D.

Define

$$\mathscr{N}(\mathbf{K}) := \{A \in \mathscr{P}(\mathbf{K}) \mid \mathrm{sv}({}^{\mathbf{K}}P^Q, A) = 0, \forall {}^{\mathbf{K}}P^Q \in {}^{\mathbf{K}}\rho^E\},$$

$$\mathscr{N}(\mathbf{L}) := \{A \in \mathscr{P}(\mathbf{L}) \mid \mathrm{sv}({}^{\mathbf{L}}P^Q, A) = 0, \forall {}^{\mathbf{L}}P^Q \in {}^{\mathbf{L}}\rho^E\},$$

$$\mathscr{R}(\mathbf{K}) := \{A \in \mathscr{P}(\mathbf{K}) \mid A = \bigcup_{s \in K_i} {}^{\mathbf{K}}A_s^{K_i} \times F_s, F_s \in \mathscr{F}_s, i \in N\},$$

$$\mathscr{R}(\mathbf{L}) := \left\{A \in \mathscr{P}(\mathbf{L}) \mid A = \bigcup_{s \in K_i} {}^{\mathbf{L}}A_s^{K_i} \times F_s, F_s \in \mathscr{F}_s, i \in N\right\}.$$

We want to find an appropriate σ-isomorphism Ψ from $\mathscr{P}(\mathbf{K})/\mathscr{N}(\mathbf{K})$ to $\mathscr{P}(\mathbf{L})/\mathscr{N}(\mathbf{L})$, and to this extent we start off with

$$\Psi: \mathscr{R}(\mathbf{K})/\mathscr{N}(\mathbf{K}) \to \mathscr{R}(\mathbf{L})/\mathscr{N}(\mathbf{L}),$$

$$\left[\bigcup_{s \in K_i} {}^{\mathbf{K}}A_s^{K_i} \times F_s\right] \mapsto \left[\bigcup_{s \in K_i} {}^{\mathbf{L}}A_s^{K_i} \times F_s\right],$$

Note that due to (6.6), $\mathscr{R}(\mathbf{L})/\mathscr{N}(\mathbf{L})$ is a Boolean ring, although $\mathscr{R}(\mathbf{L})$ does not have to be a ring.

LEMMA 6.9. $\Psi: \mathscr{R}(\mathbf{K})/\mathscr{N}(\mathbf{K}) \to \mathscr{R}(\mathbf{L})/\mathscr{N}(\mathbf{L})$ is a σ-isomorphism of Boolean rings such that

$$\Psi' \, {}^{\mathbf{L}}p^Q \left[\bigcup_{s \in K_i} {}^{\mathbf{K}}A_s^{K_i} \times F_s \right] = {}^{\mathbf{K}}p^Q \left[\bigcup_{s \in K_i} {}^{\mathbf{K}}A_s^{K_i} \times F_s \right], \tag{6.11}$$

$$\mathrm{sv}\left({}^{\mathbf{L}}p^Q, \bigcup_{s \in K_i} {}^{\mathbf{L}}A_s^{K_i} \times F_s \right) = \mathrm{sv}\left({}^{\mathbf{K}}p^Q, \bigcup_{s \in K_i} {}^{\mathbf{K}}A_s^{K_i} \times F_s \right), \tag{6.12}$$

$\forall {}^{\mathbf{L}}p^Q \in {}^{L}\rho^E$.

Proof. Let us first show that Ψ is uniquely defined. The set $\Delta_m := \{\delta_{u,\omega} \mid \delta_{u,\omega}$ is a measure on $(S(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K}))$ with $\delta_{u,\omega} \neq 0$, $\mathrm{sv}(\delta_{u,\omega}) < \infty$, $\mathrm{supp}(\delta_{u,\omega}) = v^{\mathbf{K}}(u) \times e_\omega^u$, where e_ω^u is the atom in $\mathscr{F}_u$ containing ω; $u \in K_m, \omega \in \Omega\}$ is contained in ${}^{\mathbf{K}}\rho^E$. So if $[{}^{\mathbf{K}}A_s^{K_n} \times F_s] = [\bigcup_{t \in K_m} {}^{\mathbf{K}}A_t^{K_m} \times G_t]$ for an $m \geqslant n$, it follows that $\delta_{u,\omega}(\bigcup_{t \in K_m} ({}^{\mathbf{K}}A_t^{K_m} \times G_t \,\triangle\, {}^{\mathbf{K}}A_s^{K_n} \times F_s)) = 0$, $\forall \delta_{u,\omega} \in \Delta_m$, and therefore $\bigcup_{t \in K_m} A_t^{K_m} \times G_t \equiv A_s^{K_n} \times F_s$. Using $(A_v^{K_i})^c = \bigcup_{u \in K_i, u \neq v}^{d} A_u^{K_i}$, (6.10) implies

$$\begin{aligned} &{}^{\mathbf{L}}p^Q \left[\left(\bigcup_{t \in K_m} {}^{\mathbf{L}}A_t^{K_m} \times G_t \right) \triangle\, {}^{\mathbf{L}}A_s^{K_n} \times F_s \cap {}^{\mathbf{L}}A_u^{L_i} \times F_u \right] \\ &= {}^{\mathbf{K}}p^Q \left[\left(\bigcup_{t \in K_m} {}^{\mathbf{K}}A_t^{K_m} \times G_t \right) \triangle\, {}^{\mathbf{K}}A_s^{K_n} \times F_s \cap {}^{\mathbf{K}}A_u^{L_i} \times F_u \right] \\ &= 0 \end{aligned}$$

and this proves

$$\Psi \left[\bigcup_{t \in K_m} {}^{\mathbf{K}}A_t^{K_m} \times G_t \right] = \Psi[{}^{\mathbf{K}}A_s^{K_n} \times F_s].$$

Ψ is surjective, injective because $[\bigcup_{t \in K_m} {}^{\mathbf{L}}A_t^{K_m} \times G_t] = [{}^{\mathbf{L}}A_s^{K_n} \times F_s]$ implies $\bigcup_{t \in K_m} A_t^{K_m} \times G_t = A_s^{K_n} \times F_s$ in the same way as above, and a homomorphism because $\psi([A]^c) = (\psi([A]))^c$ by (6.6). If $[A_i] \uparrow [A]$ in $\mathscr{R}(\mathbf{K})/\mathscr{N}(\mathbf{K})$, (6.10) proves

$$\begin{aligned} &{}^{\mathbf{L}}p^Q \left[\left(\Psi[A] \setminus \bigcup_{i \in \mathbb{N}} \Psi[A_i] \right) \cap {}^{\mathbf{L}}A_u^{L_j} \times F_u \right] \\ &= \lim_{i \to \infty} {}^{\mathbf{K}}p^Q[([A] \setminus [A_i]) \cap {}^{\mathbf{K}}A_u^{L_j} \times F_u] \\ &= 0, \end{aligned}$$

$\forall^{\mathbf{L}}p^Q \in {}^{\mathbf{L}}\rho^E$, i.e., $\Psi([A_i]) \uparrow \Psi([A])$, and Ψ is a σ-isomorphism. Equations (6.11) and (6.12) follow from (6.5) and (6.10), respectively. Q.E.D.

LEMMA 6.10. *The map Ψ has an extension to a σ-isomorphism from $\mathscr{P}(\mathbf{K})/\mathscr{N}(\mathbf{K})$ to $\mathscr{P}(\mathbf{L})/\mathscr{N}(\mathbf{L})$ such that $\Psi'\ {}^{\mathbf{L}}p^Q = {}^{\mathbf{K}}p^Q$, $\forall^{\mathbf{L}}p^Q \in {}^{\mathbf{L}}\rho^E$.*

Proof. Let $\mathbb{O}$ be the class of ordinal numbers and define

$$\mathscr{A}_0(\cdot) := \mathscr{R}(\cdot)/\mathscr{N}(\cdot),$$

$$\mathscr{A}_\iota(\cdot) := \{A \in \mathscr{P}(\cdot)/\mathscr{N}(\cdot) \mid \exists (A_i)_{i\in\mathbb{N}} \subseteq \mathscr{A}_{\iota-1} \text{ with } A_i \uparrow A\}$$

if $\iota = \iota_0 + n$, ι_0 a limit point, including 0, and n odd,

$$\mathscr{A}_\iota(\cdot) := \{A \in \mathscr{P}(\cdot)/\mathscr{N}(\cdot) \mid \exists (A_i)_{i\in\mathbb{N}} \subseteq \mathscr{A}_{\iota-1} \text{ with } A_i \downarrow A\}$$

if $\iota = \iota_0 + n$, ι_0 a limit point, including 0, and $n \neq 0$ even, and

$$\mathscr{A}_\iota(\cdot) := \bigcup_{\iota' < \iota} \mathscr{A}_{\iota'}(\cdot)$$

if $\iota \neq 0$ is a limit point. To prove that Ψ has an extension, which we will also denote by Ψ, we will use a transfinite induction argument with respect to ι. Let us assume first that $\iota = \iota_0 + n$ with n odd, and let $\psi : \mathscr{A}_{\iota-1}(\mathbf{K}) \to \mathscr{A}_{\iota-1}(\mathbf{L})$ be a bijection such that

$$\begin{aligned}
&\psi(A \cup B) = \psi(A) \cup \psi(B),\\
&\psi(A \cap B) = \psi(A) \cap \psi(B), \qquad \forall A, B \in \mathscr{A}_{\iota-1}(\mathbf{K});\\
&\psi(A^c) = (\psi(A))^c, \qquad \forall A \in \bigcup_{\iota' < \iota - 1} \mathscr{A}_{\iota'}(\mathbf{K}); \qquad (6.13)\\
&{}^{\mathbf{K}}p^Q[A \cap {}^{\mathbf{K}}A_s^{L_j} \times F_s] = {}^{\mathbf{L}}p^Q[\Psi(A) \cap {}^{\mathbf{L}}A_s^{L_j} \times F_s]\\
&\qquad\qquad \forall^{\cdot}p^Q \in {}^{\cdot}\rho^E, \qquad \forall A \in \mathscr{A}_{\iota-1}(\mathbf{K}).
\end{aligned}$$

If $A \in \mathscr{A}_\iota(\mathbf{K})$ and $(A_i)_{i\in\mathbb{N}} \subseteq \mathscr{A}_{i-1}(\mathbf{K})$ with $A_i \uparrow A$, then define $\Psi(A) := \bigcup_{i\in\mathbb{N}} \Psi(A_i)$. This definition is unique: for if $(A_i)_{i\in\mathbb{N}}, (B_i)_{i\in\mathbb{N}} \subseteq \mathscr{A}_{\iota-1}(\mathbf{K})$ with $A_i \uparrow A$ and $B_i \uparrow A$, then (6.13) implies

$$\begin{aligned}
&\lim_{m\to\infty} {}^{\mathbf{L}}p^Q \left[\left(\bigcup_{i\leqslant m} \Psi(A_i \cup B_i) \setminus \bigcup_{i\leqslant m} \Psi(B_i)\right) \cap {}^{\mathbf{L}}A_s^{L_j} \times F_s\right]\\
&\quad = \lim_{m\to\infty} {}^{\mathbf{K}}p^Q \left[\left(\bigcup_{i\leqslant m} (A_i \cup B_i) \setminus \bigcup_{i\leqslant m} B_i\right) \cap {}^{\mathbf{K}}A_s^{L_j} \times F_s\right] = 0,
\end{aligned}$$

$\forall^{\cdot}p^Q \in {}^{\cdot}\rho^E$, so that $\mathrm{sv}({}^{\mathbf{L}}p^Q, (\bigcup_{i\in N} \Psi(A_i)) \Delta (\bigcup_{i\in N} \Psi(B_i))) = 0$, $\forall^{\mathbf{L}}p^Q \in {}^{\mathbf{L}}\rho^E$, and therefore $\bigcup_{i\in\mathbb{N}} \Psi(A_i) = \bigcup_{i\in\mathbb{N}} \Psi(B_i)$. The same type of argument proves that Ψ satisfies (6.13) on $\mathscr{A}_\iota(\cdot)$.

The case $\iota = \iota_0 + n$, $n \neq 0$ even, is of course the same, and if ι is a limit point, $\psi : \mathscr{A}_\iota(\mathbf{K}) \to \mathscr{A}_\iota(\mathbf{L})$ defined by its values on $\mathscr{A}_{\iota'}(\cdot)$, $\iota' < \iota$, is easily seen to satisfy (6.13) as well.

Let ω be the first uncountable ordinal number; obviously $\mathscr{A}_\omega(\mathbf{K}) = \mathscr{P}(\mathbf{K})/\mathscr{N}(\mathbf{K})$. But we also have $\mathscr{A}_\omega(\mathbf{L}) = \mathscr{P}(\mathbf{L})/\mathscr{N}(\mathbf{L})$: for if $s \in \bigcup_{j\in\mathbb{N}} L_j$ and $\nu^{\mathbf{K}}(s) = (s_i)_{i\in\mathbb{N}}$, (6.9) proves

$$\lim_{i\to\infty} {}^{\mathbf{L}}P^Q[(\{x \geqslant \nu^{\mathbf{L}}(s_i)\}\backslash\{x \geqslant \nu^{\mathbf{L}}(s)\}) \times F \cap {}^{\mathbf{L}}A_u^{L_j} \times F_u]$$

$$= \lim_{i\to\infty} {}^{\mathbf{K}}P^Q[(x \geqslant \nu^{\mathbf{K}}(s_i)\}\backslash\{x \geqslant \nu^{\mathbf{K}}(s)\}) \times F \cap {}^{\mathbf{K}}A_u^{L_j} \times F_u]$$

$$= 0,$$

$\forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}_{s_i}$, so that $[\{x \geqslant \nu^{\mathbf{L}}(s)\} \times F] \in \sigma(\mathscr{R}(\mathbf{L})/\mathscr{N}(\mathbf{L}))$, and these sets generate $\mathscr{P}(\mathbf{L})/\mathscr{N}(\mathbf{L})$. The lemma follows. Q.E.D.

This proves Theorem 6.6 for ${}^{\cdot}\rho^E$. Because $\mathscr{N}(\cdot, {}^{\cdot}\chi) := \{A \in \mathscr{P}(\cdot) \mid \mathrm{sv}({}^{\cdot}P^X, A) = 0,\ \forall\, {}^{\cdot}P^X \in {}^{\cdot}\chi\} \supseteq \mathscr{N}(\cdot)$, Ψ induces a σ-isomorphism

$$\begin{aligned}\hat{\Psi} : \mathscr{P}(\mathbf{K})/\mathscr{N}(\mathbf{K})\big/_{\mathscr{N}(\mathbf{K},\, {}^{\mathbf{K}}\chi)} &\cong \mathscr{P}(\mathbf{K})/\mathscr{N}(\mathbf{K}, {}^{\mathbf{K}}\chi) \\ \to \mathscr{P}(\mathbf{L})/\mathscr{N}(\mathbf{L})\big/_{\mathscr{N}(\mathbf{L},\, {}^{\mathbf{L}}\chi)} &\cong \mathscr{P}(\mathbf{L})/\mathscr{N}(\mathbf{L}, {}^{\mathbf{L}}\chi)\end{aligned}$$

with $\hat{\Psi}'\,{}^{\mathbf{L}}P^X = {}^{\mathbf{K}}P^X$. The same argument proves Theorem 6.6 for ${}^{\cdot}\beta^E$ and ${}^{\cdot}\rho^+$ (noting that $\mathbf{K}$–var$(Q) = \mathbf{L}$–var(Q) and ${}^{\mathbf{K}}P^Q \geqslant 0$ iff ${}^{\mathbf{L}}P^Q \geqslant 0$ due to (6.8)), and also Corollary 6.7. Q.E.D.

Let

$$\mathscr{N}^S(\cdot) = \{A \in S(\cdot) \mid \mathrm{sv}({}^{\cdot}P^f, A) = 0,\ \forall \text{ regular } f \text{ with } \cdot\text{-sv}(f) < \infty\}$$

and note that if $A \in \mathscr{N}^S(\cdot)$, then $A \times \Omega \in \mathscr{N}(\cdot)$, as can be seen by looking at the regular maps $f(s) := Q_s(\Omega)$ for ${}^{\cdot}P^Q \in {}^{\cdot}\rho^E$.

THEOREM 6.11. *There exist sets* $N(\cdot) \in \mathscr{N}^S(\cdot)$ *and a map* $\Theta : S'(\mathbf{K}) \times \Omega := (S(\mathbf{K})\backslash N(\mathbf{K})) \times \Omega \to S'(\mathbf{L}) \times \Omega := (S(\mathbf{L})\backslash N(\mathbf{L})) \times \Omega$ *such that*

(i) Θ *is an isomorphism of the measurable spaces* $(S'(\mathbf{K}) \times \Omega, \mathscr{P}(\mathbf{K})|_{S'(\mathbf{K})\times\Omega})$ *and* $(S'(\mathbf{L}) \times \Omega, \mathscr{P}(\mathbf{L})|_{S'(\mathbf{L})\times\Omega})$.

(ii) $\Theta^*\, {}^{\mathbf{K}}P^Q = {}^{\mathbf{L}}P^Q \quad \forall\, {}^{\mathbf{K}}P^Q \in {}^{\mathbf{K}}\rho^E$, *where*

$$\Theta^*\, {}^{\mathbf{K}}P^Q[A] := \begin{cases} {}^{\mathbf{K}}P^Q[\Theta^{-1}(A)], & A \in \mathscr{P}(\mathbf{L})|_{S'(\mathbf{L})\times\Omega} \\ 0, & A \in \mathscr{P}(\mathbf{L})|_{N(\mathbf{L})\times\Omega}.\end{cases}$$

In particular, Θ^* defines a bijection between the sets ${}^{\cdot}\rho^E$, ${}^{\cdot}\beta^E$, ${}^{\cdot}\rho^+$ and ${}^{\cdot}\chi$ respectively.

Proof. $(S(\cdot), \mathscr{S}(\cdot))$ are standard Borel spaces, and since they are separable, they can be considered as complete separable metric spaces furnished with their Borel-σ-algebras. Theorem 6.6 yields a σ-isomorphism $\Psi : S(\mathbf{K})/\mathscr{N}^S(\mathbf{K}) \to S(\mathbf{L})/\mathscr{N}^S(\mathbf{L})$, so by Theorem 12 of [23, p. 329], there exist sets $N(\cdot) \in \mathscr{N}^S(\cdot)$ and an isomorphism $\tilde{\Theta}$ from $(S'(\mathbf{K}) := S(\mathbf{K})\backslash N(\mathbf{K}), \mathscr{S}(\mathbf{K})|_{S'(\mathbf{K})})$ to $(S'(\mathbf{L}) := S(\mathbf{L})\backslash N(\mathbf{L}), \mathscr{S}(\mathbf{L})|_{S'(\mathbf{L})})$ such that $\Psi[A] = [\tilde{\Theta}(A)]$. This proves the theorem for $|\Omega| = 1$. From the construction of Ψ we know that

$$\begin{aligned}&\tilde{\Theta}(\{x \geqslant v^{\mathbf{K}}(s^1),..., v^{\mathbf{K}}(s^m)\} \cap S'(\mathbf{K}))\\&\quad = \{x \geqslant v^{\mathbf{L}}(s^1),..., v^{\mathbf{L}}(s^m)\} \cap S'(\mathbf{L}) \qquad \operatorname{mod} \mathscr{N}^S(\mathbf{L}) \qquad (6.14)\end{aligned}$$

for $\{s^1,..., s^m\} \subseteq V$. Define

$$\begin{aligned}{}^{\mathbf{K}}A_{s^1,...,s^m} &:= \{x \geqslant v^{\mathbf{K}}(s^1),..., v^{\mathbf{K}}(s^m)\} \cap S'(\mathbf{K})\\&\qquad \Delta\tilde{\Theta}^{-1}(\{x \geqslant v^{\mathbf{L}}(s^1),..., v^{\mathbf{L}}(s^m)\} \cap S'(\mathbf{L}));\\{}^{\mathbf{L}}A_{s^1,...,s^m} &:= \tilde{\Theta}({}^{\mathbf{K}}A_{s^1,...,s^m});\\{}^{\cdot}A &:= \bigcup\left({}^{\cdot}A_{s^1,...,s^m} \,\middle|\, s^i \in \left(\bigcup_{j\in\mathbb{N}} K_j\right)\right.\\&\qquad \left.\cup \left(\bigcup_{j\in\mathbb{N}} L_j\right),\ 1 \leqslant i \leqslant m,\ m \in \mathbb{N}\right).\end{aligned}$$

${}^{\cdot}A \in \mathscr{N}^S(\cdot)$, and $\tilde{\Theta}$ restricted to $(S''(\cdot) := S'(\cdot)\backslash {}^{\cdot}A, \mathscr{S}(\cdot)|_{S''(\cdot)})$ is an isomorphism with the property that

$$\begin{aligned}&\tilde{\Theta}(\{x \in S''(\mathbf{K}) \mid x \geqslant v^{\mathbf{K}}(s^1),..., v^{\mathbf{K}}(s^m)\})\\&\quad = \{x \in S''(\mathbf{L}) \mid x \geqslant v^{\mathbf{L}}(s^1),..., v^{\mathbf{L}}(s^m)\} \qquad (6.15)\end{aligned}$$

for $s^i \in (\bigcup_{j\in\mathbb{N}} K_j) \cup (\bigcup_{j\in\mathbb{N}} L_j)$. If $\Theta : S''(\mathbf{K}) \times \Omega \to S''(\mathbf{L}) \times \Omega$ is defined by $\Theta(s, \omega) := (\tilde{\Theta}(s), \omega)$, then Θ is bijective, and Θ as well as Θ^{-1} are measurable due to (6.15). $\Theta^* \, {}^{\mathbf{K}}P^Q = {}^{\mathbf{L}}P^Q$ follows from (6.14) and the sieve formula. Q.E.D.

Remark. There are counterexamples showing Theorems 6.6, 6.7 and 6.11 to be wrong if condition (6.2) is omitted in the definition of a regular E-quasimartingale, or is replaced by $Q_s(F) = \text{weak-}\lim_{s_i \uparrow s} Q_{s_i}(F)\ \forall F \in \bigcup_{i\in\mathbb{N}} \mathscr{F}_{s_i}$.

APPENDIX: THE MÖBIUSINVERSION

Let V be a finite partially ordered set.

DEFINITION A.1. The Möbiusfunction $\mu : V \times V \to Z$ is defined recursively by

$$\mu(s,s) := 1, \qquad \forall s \in V,$$

$$\mu(s,t) := \begin{cases} -\sum\limits_{s \leqslant r < t} \mu(s,r), & \forall t \in V, t > s \\ 0, & \forall t \in V, t \not\geqslant s. \end{cases}$$

Equivalently, we could define $\mu(t,t) := 1$ $\forall t \in V$, $\mu(s,t) := -\sum_{s < r \leqslant t} \mu(r,t)$ $\forall s \in V$, $s < t$, and $\mu(s,t) = 0$ otherwise.

LEMMA A.2 (Möbiusversion). *Let $f, g : V \to G$ be maps with values in a commutative group with the group operation $+$. Then*

$$f(s) = \sum_{r \geqslant s} g(r), \forall s \in V \Leftrightarrow g(r) = \sum_{t \geqslant r} \mu(r,t) f(t), \forall t \in V,$$

$$f(s) = \sum_{r \leqslant s} g(r), \forall s \in V \Leftrightarrow g(r) = \sum_{t \leqslant r} \mu(t,r) f(t), \forall t \in V.$$

Here, for $n \in \mathbb{N}$ and $s \in G$, $n \cdot s$ is to be read as $s + s + \cdots + s$ (n times). The proof follows by inserting one side in the other and then using $\sum_{r: r \geqslant s} \mu(r,t) = \sum_{r: r \leqslant t} \mu(s,r) = 1$ if $s = t$, and 0 otherwise.

The operator $D^{V,\geqslant}$ defined by $D_s^{V,\geqslant} f := \sum_{r \geqslant s} \mu(s,r) f(r) = \sum_{r \in V} \mu(s,r) f(r)$ is called the upper difference operator, and $D^{V,\leqslant}$ defined by $D_s^{V,\leqslant} f := \sum_{r \leqslant s} \mu(r,s) f(r) = \sum_{r \in V} \mu(r,s) f(s)$ the lower difference operator. The only special result about difference operators we have used was the comparison of $D^{V,\geqslant}$ and $D^{W,\geqslant}$ for a subset W of V containing 0 (which we assume to exist). For $s \in W$ denote $\{x \in V \mid x \geqslant s, x \not\geqslant r, \forall r \in W \text{ with } r > s\}$ by A_s^W and define a map $\tau : V \to W \subseteq V$ by $\tau(t) := s$ iff $t \in A_s^W$. In the theory of combinatorics, τ is called a coclosure.

LEMMA A.3. $D_s^{W,\geqslant} f = \sum_{t \in V, \tau(t) = s} D_t^{V,\geqslant} f$, $\forall s \in W$.

This follows from $f(s) = \sum_{r \in W, r \geqslant s} (\sum_{t \in A_r} D_t^{V,\geqslant}) = \sum_{r \in W, r \geqslant s} D_r^{W,\geqslant} f$ and Lemma A.2.

As an example, let $V = \{A \mid A \subseteq \{1, \ldots, n\}\}$. Then $\mu(A,B) = (-1^{|A| - |B|}$ if $A \subseteq B$, and 0 otherwise, and therefore

$$D_\varnothing^{V,\geqslant} f = f(\varnothing) - \sum_{|A|=1} f(A) + \sum_{|A|=2} f(A) - \cdots + (-1)^n f(\{1, \ldots, n\}).$$

An important application of this example is:

THEOREM A.4 (Principle of inclusion-exclusion, sieve formula). *Let $\mathscr{A}$ be a finite ring, $A_1, \ldots, A_m \in \mathscr{A}$. If $f: \mathscr{A} \to G$ is a map with $f(\varnothing) = 0$ and $f(A \cup B) + f(A \cap B) = f(A) + f(B)\ \forall A, B \in \mathscr{A}$, then*

$$f\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} f(A_i) - \sum_{1 \leqslant i < j \leqslant m} f(A_i \cap A_j) + \cdots + (-1)^{m-1} f\left(\bigcap_{i=1}^{m} A_i\right).$$

For further details and examples, see e.g. [1, 5, 22].

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